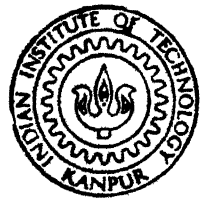
BIFURCATION THEORETIC APPROACH TO LARGE AMPLITUDE OSCILLATIONS OF STRETCHED STRINGS

by

JYOTI SWARUP



DEPARTMENT OF MECHANICAL ENGINEERING

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JYOTI SWARUP

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CERTIFICATE

This is to certify that the thesis titled " BIFURCATION THEORETIC APPROACH TO LARGE AMPLITUDE OSCILLATIONS OF STRETCHED STRINGS " is a record of work carried out by Mr. Jyoti Swarup under our supervision and has not been submitted elsewhere for a degree.

B. L. Dhoopar
Professor
Department of Mechanical Engineering
Indian Institute of Technology
Kanpur.

MSKalra

M. S. KALRA
Assistant Professor
Department of Mechanical Engineering
Indian Institute of Technology
Kanpur.

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Jyoti Swarup

CONTENTS

ABSTRACT

Large amplitude oscillations of strings under forcing induced by transverse loading and the axial movement of supports, exhibit several features near their resonance frequencies. The response characteristics of such multi-parameter systems undergo drastic changes as the parameters are varied in the neighborhood of a certain set of their values. For these values of parameters, the governing equation of motion is singular. Under such conditions more than one set of response characteristics coincide. The system follows the one which is stable. This information is represented graphically as bifurcation diagram. Bifurcation diagrams are plots between the response variables and the parameters.

In this work, an attempt has been made to study the stationary solutions for the system when transverse forcing is near a resonance frequency, and movement of supports is done in the neighborhood of twice that frequency. The bifurcation diagrams for the conditions for which the governing equations are not singular have been determined by direct inversion of the equations. This procedure fails as the parameters approach the closed set of values for which the evolution equation is singular. For such cases, the

singularities of the problem are classified on the basis of their bifurcation characteristics. This property helps in representing the possible bifurcations around these critical values in minimal terms and also is of adequate use in the study of multi-parameter systems.

Chapter 1

INTRODUCTION

The analysis of non-linear evolutionary problems is quite critical for physical systems, mechanical or otherwise. Non-linear equations are frequent in mechanical systems whenever an attempt is made to model the systems more precisely. Even in the domain of linear physical relations, like stress-strain relationship, geometrical and other system specifications coupled with Newton's laws of motion lead to a set of non-linear evolution equations.

As typical of even simple non-linear equations, non-linear models may predict one or more steady state solutions. These solutions correspond to the system configurations, after lapse of reasonably large time interval. If certain parameters are to be varied in the equations of motion, there may be even more complexities such as new emerging steady state solutions. In physical sense it may become possible for the system to change its response characteristics. The physical systems, when confront such situation, follow the solution which shows temporal stability. In other words, those solutions are prefered to which all the transient solutions in the neighborhood converge with passage of time. Hence, study of such situations, when many new steady solutions emerge from one such solution on variation of parameters, are critical to decide the actual response of the system.

In the present work an attempt has been made to analyse the equations of motion of metallic string under transverse and axial loading. The response of a string to such forcings is the simplest

example of continuous systems exhibiting characteristics not explainable by the linear theory.

1.1 *EXPERIMENTAL RESULTS FOR STRING UNDER TRANSVERSE LOADING*

It was first reported by *Harrison (1948)* that when a metallic string is placed in a magnetic field and alternating current is passed through it, even though excitation is always planar (perpendicular to the magnetic field) conditions exist under which the planar motion becomes unstable and wire begins to whirl.

This phenomenon is a consequence of an internal resonance due to presence of non-linear coupling terms in the equations of motion. Due to symmetry of the system, the frequency of motion in the plane perpendicular to the plane of excitation is same as that in the plane of excitation. Thus, the two motions are strongly coupled.

The behavior of strings under different sets of conditions has been an area of much experimental interest. *Lee (1957)*, *Ramakrishna* and *Murthy (1965)*, *Kahut* and *Mathews (1971)* and *Eller (1972)* conducted experiments under different sets of conditions and reported phenomena of jump and whirling.

General observation of the experiments conducted on the strings under planar excitation is that, response of string is planar provided the response amplitude is smaller than a critical value. Above this value the planar motion ceases to be stable and gives way to whirling motion.

As the frequency is varied from a value far below the natural frequency to a value far above, the observed response of the string is at first confined to the plane of driving force, but as soon as

amplitude of response exceeds a certain value, the planar motion becomes unstable and the string acquires an out-of-plane motion.

As the frequency is increased further, the amplitudes of both in-plane and out-of-plane motions increase until a critical frequency is reached at which the out-of-plane motion disappears and the amplitude of in-plane motion simultaneously becomes smaller.

On the other hand, if the frequency is decreased from a value far above resonance frequency, the response is first confined to the plane of excitation until a critical frequency is reached at which the response amplitude becomes much larger and response string is no longer confined to the plane of excitation. As the frequency of excitation is decreased further the response amplitude decreases, and when it goes below a certain critical value the out-of-plane component vanishes.

The frequency at which the upward jump takes place is less than that at which the downward jump takes place. This indicates the hardening of string due to non-linearities.

1.2 *REVIEW OF MATHEMATICAL MODELLING EFFORTS*

Attempts have been made by the researchers to model the motion of the string incorporating such behaviors. *Anand (1966)* studied the problem incorporating viscous energy dissipation. His analysis showed that damping is critical for any such model. Only when damping is taken into account, it is possible to predict the frequency at which resonance or jump occurs.

Incorporating viscous damping in the equation of motion provides an important conclusion that resonance frequency of a

non-linear string is a function of driving force. Resonance frequency increases with the driving force. Increasing non-linearity of the string leads to increase in the resonance frequency and decrease in the response amplitude.

The effort in this work is to find and analyse the more complex phenomena that may occur as a result of combination of transverse forcing , and parameteric excitation due to movement of supports, in metallic strings.

1.3 *SYNOPSIS OF MATHEMATICAL PROCEDURES AND TECHNIQUES FOR ANALYSIS OF CONTINUOUS SYSTEMS*

Study of large amplitude vibration of continuous systems provides clues to their response characteristics, which are not explained by the linear theory. Strings, beams and tubes are in a way similar as they can be catagorised as one dimensional continuous systems. They show behavior quite similar to those discussed in the previous section.

Plates under excitation also show response characteristics which are not explained by the linear models. Flutter is one such phenomenon in which two dimensional continuous objects interact with fluid flow.

Critical nature of response of non-linear systems pose significant problem in design, especially with the design of dynamic systems. Stability analysis is an integral part of any such effort.

For systems governed by non-linear evolution equations, exact solutions are not generally obtainable, hence recourse is to approximate analysis. The methods used for analysis generally

consist of purely analytical techniques, purely numerical techniques, and numerical perturbation techniques. The purely analytical techniques are only applicable to the systems with simple geometries and boundary conditions. Purely numerical techniques such as finite difference analysis or finite element analysis require large computation capacity and time, and do not bear much insight about general nature of response of the systems.

Analysis in this work consists of numerical perturbation techniques and attempts have been made to analyse the problem taking some ideas from *BIFURCATION THEORY* and *SINGULARITY THEORY*.

As the non-linear problems do not allow exact solutions recourse is often, to try to study the solution in which some characteristics remain invariant with time. Physically, this corresponds to analysis of the general behavior of the systems over a large time interval. These characteristics may be coordinate invariance in some linearly transformed domain (stationary solutions), periodicity or a combination of both. *BIFURCATION THEORY* is a theory of such equilibrium solutions. The theory discusses the methodology by which such solutions may be identified, or the conditions required for existence and stability of such solutions may be prescribed.

In non-linear evolution problems of many variables and parameters the solutions in themselves contain little information about general nature of the system, it is the interaction between the solutions of different types that provide clues to the general behavior. Such interactions are possible at the *singular points* i.e., the points, where the solutions intersect or linear terms in the determining equations vanish and the solutions around these are determined by non-linear terms.

For this reason it makes sense to classify the solutions by

their organisational properties such as type of bifurcation that the solution is capable of displaying with variation of parameters. Based on such a classification, implications of variation of a parameter, or a combination, on the nature of the response may be predicted. This process of classification comes in domain of *SINGULARITY THEORY.*

1.4 *GENERAL NATURE OF CONTINUOS SYSTEMS*

Continuous systems such as strings, beams and plates are known to show certain phenomena, as whirling in strings, beams and tubes, and flutter in case of plates. These phenemena are not explained by linearised equation of motion. Non-linear behavior of response in such cases can be attributed to large amplitude vibration in the system.

For small amplitude oscillations, response can be adequately described by linear equations and boundary conditions. At higher amplitude the non-linear behavior becomes more prominent. Whirling and jump occur only at amplitude more than a certain value. Motion evolution equations modelled on large amplitude deformation in such systems have higher order non-linear coupling terms which become significant at large displacement.

The source of such non-linearity in the string is the geometry of the system. At higher amplitude stretching of wire becomes a non-linear phenomena and leads to a non-linear relationship between strain and displacement.

To facilitate the analysis the non-linear couplings are considered to be a perturbation from the linearised system. The solutions of the linear equations are used as spatial variation. Using orthogonality of the mode shapes, a non-linear coupled

second order differential equation is obtained. This describes the temporal behavior of the system.

The latter equations are reduced to a set of ordinary differential equation using a perturbation technique such as the method of multiple scales. The equations so obtained can be proved to be an approximate description of the actual physical process.

1.5 *THE METHOD OF MULTIPLE SCALES*

The technique of multiple scales [2] consists of viewing the solution which is a function of one variable (i.e. time, t), as a function of many variables, namely (t, ϵt, $\epsilon^2 t$,.....,$\epsilon^n t$), where ϵ is a small positive number. In this method the expansion representing the response is considered to be a function of multiple independent variables, or scales, instead of a single variable. the process starts with introduction of new independent variables given by,

$$T_n = \epsilon^n t \qquad \text{for } n = 0, 1, 2, \ldots\ldots \tag{1.01}$$

Thus, the derivatives with respect to t, becomes expansions in terms of the partial derivatives with respect to T_n.

$$\frac{d}{dt} = \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} + \ldots\ldots = D_0 + \epsilon D_1 + \ldots\ldots \tag{1.02}$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 (D_1^2 + 2D_0 D_1) + \ldots\ldots$$

The solutions are assummed to be expansions of form,

$$x(t : \epsilon) = \epsilon x_1(T_0, T_1, T_2,\ldots\ldots) + \epsilon^2 x_2(T_0, T_1, T_2,\ldots\ldots) \tag{1.03}$$

In this expansion the number of independent time scales needed depends on the order to which the expansion is carried out. If the expansion is carried out to order of $O(\epsilon^2)$, then T_0 and T_1 are needed. For example, consider the system governed by equation of the type

$$\ddot{x} + \sum_{n=1}^{N} \alpha_n x^n = 0 \qquad (1.04)$$

The expansion carried out to $O(\epsilon^3)$ requires the first three terms of the expansion for $\frac{d^2}{dt^2}$ and $x(t; \epsilon)$. Substituting the series in the equation and putting the coefficients of ϵ, ϵ^2, ϵ^3 to be zero so as to satisfy the equation upto order ϵ^3, we get

$$O(\epsilon): \quad D_0^2 x_1 + \omega_0^2 x_1 = 0$$

$$O(\epsilon^2): \quad D_0^2 x_2 + \omega_0^2 x_2 = -2D_0 D_1 x_1 - \alpha_2 x_1^2 \qquad (1.05)$$

$$O(\epsilon^3): \quad D_0^2 x_3 + \omega_0^2 x_3 = -2D_0 D_1 x_2 - D_1^2 x_1 - 2D_0 D_2 x_1 - 2\alpha_2 x_1 x_2 - \alpha_3 x_1^3$$

The next step is to substitute the solution of equation (1.05.i) in equation (1.05.ii) and to look for secular or resonant terms in the resulting equation and equate them to zero. This process makes x_2 to always be of order ϵx_1 as, the solution of the resulting equation with all the resonant terms zero, will always be bounded.

This reduces the need to study the resulting equation as it has no bearing on the general non-linear behavior of the solution. Solving this equation would obviously help in increasing the accuracy of the prediction.

The coefficients of the resonant terms when equated to zero give an evolution equation for x_1 in terms of large time scale T_1.

The evolution equations so obtained govern the variable of order ϵ in large time scale. These equations hold clues for non-linear behavior which occurs in the system.

1.6 *STATIONARY SOLUTIONS OF EVOLUTION EQUATIONS*

The evolution equations in the large time scale for response of order ϵ can be represented in following form.

$$\frac{dx}{dt} = F(x,\lambda) \tag{1.06}$$

For steady state solutions of coordinate invariant type, these equation symbolise a mapping from one finite dimensional space to another.

$$\begin{aligned} F &: \mathbb{X} \times \Lambda \longrightarrow \mathbb{V} \\ F_\lambda &: \mathbb{X} \longrightarrow \mathbb{V} \end{aligned} \tag{1.07}$$

The stationary solution of the equation (1.06), for any fixed value of λ , can now be observed to be the set of intersection of the graph of the map (1.07.ii) with V = 0 in $\mathbb{X}$ x $\mathbb{V}$. The parameterisation of the set by the set of parameters (λ) is represented graphically as bifurcation diagram.

As the aim is to construct and study such diagram to predict the response characteristics of the system as they change with the variation of the parameters, it is of interest to study the solution set in ($\mathbb{X}$ x Λ) about a point (0,0).

The Taylor series expansion for any such mapping with $\| x \| \lesssim \delta$, $(0 < \delta << 1)$ corresponds to

$$V = D_x F(0,\lambda)x + D_x^2 F(0,\lambda)\frac{(x,x)}{2} + \cdots \cdots \cdots \tag{1.08}$$

For the cases when $D_x F(0,0)$ is invertible, introducing a coordinate transformation.

$$U = \left[D_x F(0,0) \right]^{-1} V \qquad (1.09)$$

Taking $||\lambda|| \approx \delta$, the series is reduced to following form.

$$U = x + G(x,\lambda) \quad , \qquad (1.10)$$

$$\text{Where} \quad \| D_x G(x,\lambda) \| \lesssim \delta$$

As for our purpose we require U = 0, hence

$$x_o + G(x_o,\lambda) = 0 \qquad (1.11)$$

This equation can be solved for x in terms of λ by a simple procedure and solution in the neighborhood of (0,0) can be approximated by a form containing λ only.

$$\begin{aligned} x_o &= - G(x_o,\lambda) \\ x_o^o &= - G(0 ,\lambda) \\ x_o^1 &= - G(x_o^o,\lambda) \\ &\cdots \cdots \cdots \cdots \cdots \cdots \cdots \\ x_o^n &= - G(x_o^{n-1},\lambda) \approx x_o(\lambda) \end{aligned} \qquad (1.12)$$

This process converges, ie, valid if x and λ are sufficiently small. This procedure leads to an approximation of the bifurcation diagram about a point (0 ,0). It is due to invertibility of $D_x F(0,0)$ that it has been possible to obtain explicit representation of response as a function of parameters. Now, consider another change of variables in the equation (1.10),

$$Y = x - x_o(\lambda) \qquad (1.13)$$

Taking $\| Y \| << 1$, following form is obtained.

$$U = Y \qquad (1.14)$$

The graph of the equation (1.08) and the map (1.14) represent the same zero set.

As an observation, it can be contended that any expansion having invertible Jacobian can be mapped on such a form (1.14). As the set of invertible matrices is open, for any small change in the parameters, the Jacobian remains invertible. Thus, in the neighborhood of $\lambda = 0$, the solution is directly parameterised by the parameter set, λ, and does not show any bifurcation.

This procedure can be used to study the bifurcation behavior for a general map. The representation (1.14) has been obtained by linear transformation on the relevent coordinate system. Hence, bifurcation properties of such representation is same as those shown by the maps it represents.

The non-linear nature of the problem allows invertible transformations in the coordinates. This property is used to reduce the problem to minimal dimensions. Analysis based on such theory helps amply in plotting the bifurcation diagram as due to non-invertibility of Jaciobian in the neighborhood of singular points, simpler computational procedures fail.

Chapter 2

FORMULATION

The equations governing the motion of string are determined by Newton's laws and, geometric and material properties of the system. Formulation of the equations of motion is based on the same presented by Nayfeh and Mook [2]. To obtain the equation of motion , we consider the equilibrium of forces on the differential element PQ (Fig. 1). At any instant of time , consider a vector field (u, v, w) which is a function of time and spatial variables. The displacements are measured from the dynamically undeformed position, and x-axis is setup along the string. The string in its undeformed position is assummed to be taut and has tension of value N_o.

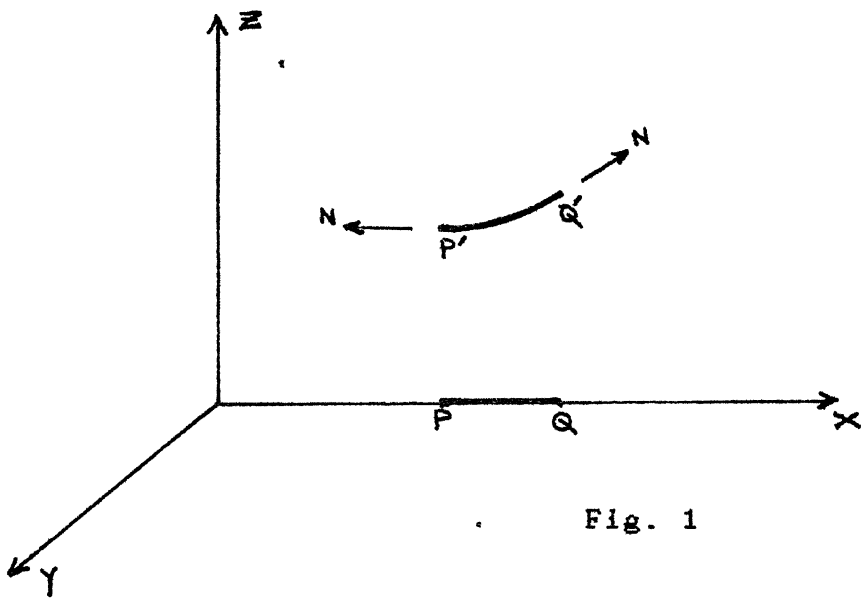

Fig. 1

The length of the differential undeformed segment of the string PQ is taken as dx. Let the coordinates of such a segment be,

$$r_p = x\, i$$
$$r_q = (x + dx)\, i \tag{2.01}$$

Then, under a displacement field (u, v, w), the displaced coordinates of the points P and Q are,

$$r_{p'} = (x + u)\, i + v j + w k$$
$$r_{q'} = (x + u + dx + u_x dx)\, i + (v + v_x dx)\, j + (w + w_x dx)\, k \tag{2.02}$$

The relative displacement vector $r_{p'q'}$ is given by,

$$r_{p'q'} = (1 + u_x)dx\, i + v_x dx\, j + w_x dx\, k \tag{2.03}$$

The length of deformed segment (PQ) at some point (t, x, y, z) is given by,

$$| r_{p'q'} | = \left[(1 + u_x)^2 + v_x^2 + w_x \right]^{\frac{1}{2}} dx = ds \tag{2.04}$$

Thus, the unit tangent vector to the deformed string segment is,

$$\delta = \left[(1 + u_x)\, i + v_x\, j + w_x\, k \right] \frac{dx}{ds} \tag{2.05}$$

The instantaneous value of tension in the deformed segment of string is then given by,

$$N = N_o + EA\left(\frac{ds - dx}{dx}\right) \tag{2.06}$$

Using the above information evolution equation of the differential segment can be formulated as,

$$\frac{\partial}{\partial x}\left[N \quad \delta \right] = \rho A \left(u_{tt} i + v_{tt} j + w_{tt} k \right) - F \qquad (2.07)$$

where,

ρ is mass per unit length

A is area of cross-section of string

F is external force per unit length acting on the differential element.

Using equation (2.05) and (2.06) , equation (2.07) can be written as,

$$\frac{\partial}{\partial x}\left[(N_o - EA)\frac{dx}{ds} + EA)((1 + u_x)\, i + v_x\, j + w_x\, k)\right]$$

$$= \rho A(u_{tt}\, i + v_{tt}\, j + w_{tt}\, k) - F \qquad (2.08)$$

In x-direction the force balance is represented by,

$$\rho A u_{tt} - EA u_{xx} = (N_o - EA)\frac{\partial}{\partial x}\left[\frac{(1 + u_x)}{[(1 + u_x)^2 + v_x^2 + w_x^2]}\right] + F_x$$

$$= (N_o - EA)\frac{\partial}{\partial x}\left[(1 + \frac{v_x^2 + w_x^2}{(1 + u_x)^2})^{-\frac{1}{2}}\right] + F_x \qquad (2.09)$$

At this instant another assumption is made, that amplitudes are small compared to the possible wave-length of the modes; hence the approximation,

$$v_x^2 \,,\; w_x^2 \ll (1 + u_x)^2 \qquad (2.10)$$

This approximation reduces the equations to following forms :

$$\rho A u_{tt} - EA u_{xx} = (N_o - EA)\frac{\partial}{\partial x}\left[-\frac{1}{2}\left[v_x^2 + w_x^2\right](1+u_x)^{-2}\right] + F_x$$
$$= \frac{1}{2}(EA - N_o)\left[\frac{\partial}{\partial x}\left[(1-2u_x)(v_x^2 + w_x^2)\right]\right] + F_x \qquad (2.11)$$

For the governing equation in a transverse direction (y),

$$\rho A V_{tt} - EA v_{xx} = (N_o - EA)\frac{\partial}{\partial x}\left[\frac{v_x}{[(1+u_x)^2 + v_x^2 + w_x^2]} - \frac{1}{2}\right]$$

$$= (N_o - EA)\left[v_{xx} - \frac{\partial}{\partial x}(v_x(u_x - u_x^2 + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2))\right] \qquad (2.12)$$

$$\rho A v_{tt} - N_o v_{xx} = (EA - N_o)\frac{\partial}{\partial x}(e v_x) + F_y$$

where, $e = u_x - u_x^2 + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2$

Similarly, for z-direction momentum balance , we have

$$\rho A w_{tt} - N_o w_{xx} = (EA - N_o)\frac{\partial}{\partial x}(e w_x) + F_z \qquad (2.13)$$

In the formulation ,

$$\frac{dx}{ds} = [(1+u_x)^2 + v_x^2 + w_x^2]^{-\frac{1}{2}}$$
$$= 1 - u_x + u_x^2 - \frac{1}{2}v_x^2 - \frac{1}{2}w_x^2 \ \cdots\cdots\cdots \qquad (2.14)$$
$$= 1 - e$$
$$e = \left(\frac{ds-dx}{ds}\right)$$

Thus, e denotes strain in the string . By symmetry of the problem e

can be taken to be a periodic function of time. Putting,

$$C_1^2 = E/\rho$$
$$C_2^2 = N_o/\rho A \qquad (2.15)$$

the set of governing equations becomes

$$u_{tt} - C_1^2 u_{xx} = (C_1^2 - C_2^2)\frac{\partial}{\partial x}\left[\left(\frac{1}{2} - u_x\right)(v_x^2 + w_x^2)\right] + F_x$$

$$v_{tt} - C_2^2 v_{xx} = (C_1^2 - C_2^2)\frac{\partial}{\partial x}\left[v_x\left(u_x - u_x^2 + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2\right)\right] + F_y \qquad (2.16)$$

$$w_{tt} - C_2^2 w_{xx} = (C_1^2 - C_2^2)\frac{\partial}{\partial x}\left[w_x\left(u_x - u_x^2 + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2\right)\right] + F_z$$

For metallic strings it is known that

$$C_1^2/C_2^2 = O(400 - 1000) \qquad (2.17)$$

We now non-dimensionalise the set of equations (2.16) and assume the forcing to be of order one. Then, the amplitudes in the transverse directions are of order one. From second equation, the amplitude of longitudinal direction oscillation is of the order C_2^2/C_1^2. This result when interpreted for the first equation means that order of magnitude of the inertial term is $1/C_1^2$ times less than that of other terms in the equation (2.16.1). Hence, due to these considerations, the longitudinal inertia term may be safely omitted.

$$-C_1^2 u_{xx} = (C_1^2 - C_2^2)\frac{\partial}{\partial x}\left[\left(\frac{1}{2} - u_x\right)(v_x^2 + w_x^2)\right] \qquad (2.18)$$

Here we assume dynamic forcing $F_x = 0$

Integrating the equation (2.18),

$$-u_x + \alpha(t) = \frac{v_x^2}{2} + \frac{w_x^2}{2} - u_x(v_x^2 + w_x^2) \qquad (2.19)$$

This equation makes it clear that u is of same order as v_x^2 and w_x^2. Hence, to the order of (v^2, w^2),

$$u_x = \alpha(t) - \frac{1}{2}(v_x^2 + w_x^2) \qquad (2.20)$$

An important implication of this result is that,

$$e(t) = \alpha(t) \qquad (2.21)$$

The equation (2.19) is integrated again, considering following boundary conditions on u,

$$u|_{x=0} = 0 \quad ; \qquad u|_{x=\ell} = \ell p(t) \quad ; \qquad (2.22)$$

where p(t) is taken to be a periodic function. The integration yields,

$$u = x\, e(t) - \frac{1}{2}\int_0^x (v_x^2 + w_x^2)\, dx \qquad (2.23)$$

Hence,

$$e(t) = p(t) + \frac{1}{2\ell}\int_0^\ell (v_x^2 + w_x^2)\, dx \qquad (2.24)$$

Adding viscous damping terms and eleminating e(t) from the transverse direction equations, the system of equation reduce

to,

$$v_{tt} + 2\mu v_t - C_2^2 v_{xx} = C_1^2 P v_{xx} + \frac{C_1^2 v_{xx}}{\ell} \int_0^\ell (v_x^2 + w_x^2)\, dx + F_y \qquad (2.25)$$

$$w_{tt} + 2\mu w_t - C_2^2 w_{xx} = C_1^2 P w_{xx} + \frac{C_1^2 w_{xx}}{\ell} \int_0^\ell (v_x^2 + w_x^2)\, dx + F_z \qquad (2.26)$$

2.1 *LINEAR MODES APPROXIMATION*

As v and w are continuous and bounded for all x, they can be approximated by following series expansion using linear modes solution as spatial variation,

$$v = \sqrt{\frac{1}{2}\ell} \sum_{n=1}^{\infty} \zeta_n(t)\, \mathrm{Sin}(n\pi x/\ell)$$

$$w = \sqrt{\frac{1}{2}\ell} \sum_{n=1}^{\infty} \eta_n(t)\, \mathrm{Sin}(n\pi x/\ell) \qquad (2.27)$$

$$\sqrt{\frac{1}{2}\ell} \sum_{n=1}^{\infty} \ddot{\zeta}_n\, \mathrm{Sin}(n\pi x/\ell) + \sqrt{\frac{1}{2}\ell} \sum_{n=1}^{\infty} \zeta_n C_2^2 n^2 \pi^2/\ell^2\, \mathrm{Sin}(n\pi x/\ell) + \sqrt{\frac{1}{2}\ell} \sum_{n=1}^{\infty} \dot{\zeta}\, \mathrm{Sin}(n\pi x/\ell)$$

$$= -C_1^2 P \sqrt{\frac{1}{2}\ell} \sum_{n=1}^{\infty} n^2 \pi^2/\ell^2\, \zeta_n \mathrm{Sin}(n\pi x/\ell) - C_1^2 \sqrt{\frac{9}{2}\ell} \sum_{n=1}^{\infty} \zeta_n n^2 \pi^2/\ell^2\, \mathrm{Sin}(n\pi x/\ell) \Bigg[$$

$$\int_0^\ell \left[\left(\sum_{m=0}^{\infty} \zeta_m m\pi/\ell\, \mathrm{Cos}(m\pi x/\ell) \right)^2 + \left(\sum_{m=0}^{\infty} \eta_m m\pi/\ell\, \mathrm{Cos}(m\pi x/\ell) \right)^2 \right] \Bigg]$$

$$+ g(x,t) \qquad (2.28)$$

Since,

$$\int_0^{\ell} \Big(\sum_{m=0}^{\infty} \zeta_m m\pi/\ell \, \mathrm{Cos}(m\pi x/\ell)\Big)^2 dx = \sum_{m=0}^{\infty} \int_0^{\ell} \zeta_m^2 m^2 \pi^2 /\ell^2 \, \mathrm{Cos}^2 (m\pi x/\ell) dx \tag{2.29}$$

As,

$$\int_0^{\ell} \mathrm{Cos}(m\pi x/\ell) \, \mathrm{Cos}(n\pi x/\ell) = \frac{\ell}{2} \delta_{mn} \tag{2.30}$$

The integral (2.29) is equal to,

$$= \sum_{m=1}^{\infty} \zeta_m^2 \frac{m^2\pi^2}{2\ell} \tag{2.31}$$

Thus, using orthogonality of linear modes, the set of equation reduces to the following form,

$$\epsilon^{\frac{1}{2}} \ell \ddot{\zeta}_n + \epsilon^{\frac{1}{2}} \ell \frac{n^2\pi^2}{2} C_2^2 \zeta_n + 2\mu_n \epsilon^{\frac{1}{2}} \ell \dot{\zeta}_n = -C_1^2 P \epsilon^{\frac{1}{2}} \ell \frac{n^2\pi^2}{2} \zeta_n$$

$$- C_1^2 \epsilon^{\frac{3}{2}} \ell^3 \zeta_n \frac{n^2\pi^4}{4\ell^4} \sum_m \big(m^2 (\zeta_m^2 + \eta_m^2) \big) + g(t) \tag{2.32}$$

Now, define some new variables ω , Γ , p denote

$$\omega_n^2 = \frac{n^2\pi^2 C_2^2}{\ell^2} \; ; \quad \Gamma = \frac{\pi^4 C_1^2}{4\ell^2} \; ; \; \epsilon p = P \frac{\pi^2 C_1^2}{\ell^2} \; ; \; \epsilon\mu_n = \mu \; ; \tag{2.33}$$

Introduction of these variables reduces equations to the form given below :

$$\ddot{\zeta}_n + \omega_n^2 \zeta_n = -\epsilon \Big[2\mu_n \dot{\zeta}_n + n^2 p \, \zeta_n + \Gamma n^2 \zeta_n \sum (m^2 (\zeta_m^2 + \eta_m^2)) \Big] + g_n(t) \tag{2.34}$$

$$\ddot{\eta}_n + \omega_n^2 \eta_n = -\epsilon \Big[2\mu_n \dot{\eta}_n + n^2 p \, \eta_n + \Gamma n^2 \eta_n \sum_m (m^2 (\zeta_m^2 + \eta_m^2)) \Big] + f_n(t)$$

2.2 *PERTURBATION ANALYSIS IN THE NEIGHBORHOOD OF A RESONANCE FREQUENCY*

Next step consists of analysing these equations in the neighborhood of resonance. As apparent from the equations, non-linear behavior becomes more prominent in its vicinty. Let,

$$\begin{aligned} g_n(t) &= 2 \in k_n \operatorname{Cos} \Omega t \\ f_n(t) &= 0 \\ p(t) &= 2 \Phi \operatorname{Cos}(2\Omega_p t + \psi) \\ \Omega &= \omega_n + \in \sigma_1 \\ \Omega_p &= 2\omega_n + \in \sigma_2 \end{aligned} \tag{2.35}$$

At this stage the method of multiple scales is applied. Consistent with the method, the solution is assumed to be an expansion in terms of the small number $\in$.

$$\begin{aligned} \zeta_n(t\ ;\in\ ) &= \zeta_{n0}(T_0, T_1) + \in \zeta_{n1}(T_0, T_1) + \cdots \cdots \\ \eta_n(t\ ;\in\ ) &= \eta_{n0}(T_0, T_1) + \in \eta_{n1}(T_0, T_1) + \cdots \cdots \end{aligned} \tag{2.36}$$

For order $\in^0$:

$$\begin{aligned} D_0^2 \zeta_{n0} + \omega_n^2 \zeta_{n0} &= 0 \\ D_0^2 \eta_{n0} + \omega_n^2 \eta_{n0} &= 0 \end{aligned} \tag{2.37}$$

The first order approximate solution is given as,

$$\begin{aligned} \zeta_{n0}(T_0, T_1) &= A_n(T_1)\exp(i\omega_n T_0) + \bar{A}_n(T_1)\exp(-i\omega_n T_0) \\ \eta_{n0}(T_0, T_1) &= B_n(T_1)\exp(i\omega_n T_0) + \bar{B}_n(T_1)\exp(-i\omega_n T_0) \end{aligned} \tag{2.38}$$

For order ϵ^1 :

$$D_0^2\zeta_{n1} + \omega_n^2\zeta_{n1} = -2D_0D_1\zeta_{n0} - 2\mu D_0\zeta_{n0} - pn^2\zeta_{n0} - n^2\Gamma\zeta_{n0}\left[\sum_m m^2(\zeta_{m0}^2 + \eta_{m0}^2)\right] + 2k_n\cos(\Omega T_0) \tag{2.39}$$

$$D_0^2\eta_{n1} + \omega_n^2\eta_{n1} = -2D_0D_1\eta_{n0} - 2\mu D_0\eta_{n0} - pn^2\eta_{n0} - n^2\Gamma\eta_{n0}\left[\sum_m m^2(\zeta_{m0}^2 + \eta_{m0}^2)\right]$$

Putting the solution (2.38) in the equations (2.39), as required by the method, following form is obtained,

$$D_0^2\zeta_{n1} + \omega_n^2\zeta_{n1} = -2D_0D_1(A_n e^{i\omega_n T_0} + \bar{A}_n e^{-i\omega_n T_0}) - 2D_0(A_n e^{i\omega_n T_0} + \bar{A}_n e^{-i\omega_n T_0})$$
$$+n^2\Phi_n(e^{i(2\omega_n+\epsilon\sigma_2)T_0+i\psi} + e^{-i(2\omega_n+\epsilon\sigma_2)T_0-i\psi})(A_n e^{i\omega_n T_0} + \bar{A}_n e^{-i\omega_n T_0})$$
$$-\Gamma n^2(A_n e^{i\omega_n T_0} + \bar{A}_n e^{-i\omega_n T_0})\left[\sum_m m^2((A_m e^{i\omega_m T_0} + \bar{A}_m e^{-i\omega_m T_0})^2\right.$$
$$\left.+(B_m e^{i\omega_m T_0} + \bar{B}_m e^{-i\omega_m T_0})^2)\right] + k_n(e^{i(\omega_n+\epsilon\sigma_1)T_0} + e^{-i(\omega_n+\epsilon\sigma_1)T_0}) \tag{2.40}$$

To eliminate secular terms from the expansion (2.40) the coefficients of $e^{i\omega_n T_0}$ and $e^{-i\omega_n T_0}$ have to be zero,

$$-2i\omega_n(A_n' + \mu A_n) - n^2\Phi_n e^{i\psi}\bar{A}_n e^{i\sigma_2 T_1} - 2\Gamma n^2 A_n\sum_m m^2(A_m\bar{A}_m + B_m\bar{B}_m)$$
$$-\Gamma n^4\bar{A}_n(A_n^2 + B_n^2) + k_n e^{i\sigma_1 T_1} = 0 \tag{2.41}$$

where ' denotes derivative with respect to T_1.

For similar reason,

Complex conjugate of the above equation = 0

For the other transverse direction,

$$-2i\omega_n(B_n' + \mu B_n) - n^2\Phi_n e^{i\psi} B_n e^{i\sigma_2 T_1} - 2\Gamma n^2 B_n \sum_m m^2 (A_m \bar{A}_m + B_m \bar{B}_m) - \Gamma n^4 \bar{B}_n (A_n^2 + B_n^2) = 0 \qquad (2.42)$$

Complex conjugate of the above equation = 0

Let,

$$A_n = \alpha_n C_n \qquad (\alpha_n \text{ is a scale factor}$$

$$B_n = \alpha_n D_n \qquad \text{to be determined later)} \qquad (2.43)$$

$$-2i\omega_n \alpha_n (C_n' + \mu C_n) - n^2 \Phi_n \alpha_n e^{i\psi} C_n e^{i\sigma_2 T_1} - 2\Gamma n^2 \alpha_n B_n \sum_m \alpha_m^2 m^2 (C_m \bar{C}_m + D_m \bar{D}_m) - \Gamma n^4 \alpha_n^3 \bar{C}_n (C_n^2 + D_n^2) + k_n e^{i\sigma_1 T_1} = 0 \qquad (2.44)$$

Let the scale factors be,

$$\alpha_n = \left(\frac{\Gamma n^4}{2\omega_n}\right)^{-\frac{1}{2}}$$

$$k = k_n/(2\omega_n)$$

$$\phi = n^2 \Phi_n e^{i\psi} / 2\omega_n \qquad (2.45)$$

$$\beta_n = \Gamma n^2/\omega_n$$

Introduction of these scale factors lead to the following set of equations,

$$-i(C_n' + \mu C_n) - \phi_n \bar{C}_n e^{i\sigma_2 T_1} - \beta_n C_n \sum_m \alpha_m^2 m^2 (C_m \bar{C}_m + D_m \bar{D}_m) - \bar{C}_n (C_n^2 + D_n^2) + k_n e^{i\omega_n T_1} = 0 \qquad (2.46)$$

$$-i(D_n' + \mu D_n) - \phi_n \bar{D}_n e^{i\sigma_2 T_1} - \beta_n D_n \sum_m \alpha_m^2 m^2 (C_m \bar{C}_m + D_m \bar{D}_m) - \bar{D}_n (C_n^2 + D_n^2) = 0$$

2.3 *EFFECT OF DAMPING*

For a mode (k) not excited either axially or transversally,

$$-i(C_n' + \mu C_n) - \phi_n \bar{C}_n e^{i\sigma_2 T_1} - \beta_n C_n \sum_m \alpha_m^2 m^2 (C_m \bar{C}_m + D_m \bar{D}_m) - \bar{C}_n (C_n^2 + D_n^2) = 0$$

$$-i(D_n' + \mu D_n) - \phi_n \bar{D}_n e^{i\sigma_2 T_1} - \beta_n D_n \sum_m \alpha_m^2 m^2 (C_m \bar{C}_m + D_m \bar{D}_m) - \bar{D}_n (C_n^2 + D_n^2) = 0 \qquad (2.47)$$

Then, multiplying equation (2.47.*i*) by $\bar{C}_k$,

multiplying complex conjugate of equation (2.47.*i*) by $-C_k$,

multiplying equation (2.47.*ii*) by $\bar{D}_k$,

multiplying complex conjugate of equation (2.47.*ii*) by $-D_k$,

and, adding the resulting equations, we have,

$$-i\left[(|C_k|^2)' + 2\mu |C_k|^2 + (|D_k|^2)' + 2\mu |D_k|^2 \right] = 0 \qquad (2.48)$$

For non-zero μ , this condition can only be satified by,

$$|C_k| = |D_k| = 0 \qquad (2.49)$$

This means that for mode (k) having no excitation the response is always zero. If excitation is restricted to a single mode (mode n), then due to damping in the system other modes are indentically zero. Due to this reason analysis can be restricted to the mode having its linear natural frequency near the frequency of forcing.

2.4 EVOLUTION EQUATIONS FOR THE FORCED MODE

let,

$$C_n = y_1 e^{i\sigma_1 T_1} \; ; \qquad D_n = y_3 e^{i\sigma_1 T_1} \; ;$$

$$\overline{C}_n = y_2 e^{-i\sigma_1 T_1} \; ; \qquad \overline{D}_n = y_4 e^{-i\sigma_1 T_1} \; ; \tag{2.50}$$

$$\theta = \sigma_2 - 2\sigma_1 \; ;$$

$$y_1' + (\mu + i\sigma_1)y_1 - (\phi i)y_2 e^{i\theta T_1} + 2\,i\,y_1(y_1 y_2 + y_3 y_4) - i\,y_2(y_1^2 + y_3^2) + i\,k = 0$$

$$y_2' + (\mu - i\sigma_1)y_2 - \overline{(\phi i)}\,y_1 e^{-i\theta T_1} + 2\,i\,y_2(y_1 y_2 + y_3 y_4) + i\,y_1(y_2^2 + y_4^2) - i\,k = 0$$

$$y_3' + (\mu + i\sigma_1)y_3 - (\phi i)y_4 e^{i\theta T_1} - 2\,i\,y_3(y_1 y_2 + y_3 y_4) - i\,y_4(y_1^2 + y_3^2) = 0$$

$$y_4' + (\mu - i\sigma_1)y_4 - \overline{(\phi i)}\,y_3 e^{-i\theta T_1} + 2\,i\,y_4(y_1 y_2 + y_3 y_4) + i\,y_3(y_1^2 + y_3^2) = 0 \tag{2.51}$$

In this work, attempt has been made to analyse the system with ($\theta = 0$), i.e., autonomus case.

As,

$$y_1 = \overline{y}_2 \quad \text{and} \quad y_3 = \overline{y}_4 \tag{2.52}$$

taking,

$$y_1 = x_1 + i x_2 \; ; \quad y_2 = x_1 - i x_2 \; ;$$

$$y_3 = x_3 + i x_4 \; ; \quad y_4 = x_3 - i x_4 \; ; \tag{2.53}$$

$$\phi = \xi\, e^{i\psi}$$

following equations are obtained,

$$x_1' + (\mu - \xi \sin(\psi))x_1 - (\sigma + \xi \cos(\psi))x_2 + x_2(3x_1^2+3x_2^2+x_3^2+3x_4^2) + 2x_3x_4x_1 = 0$$

$$x_2' + (\mu + \xi \sin(\psi))x_2 - (-\sigma + \xi \cos(\psi))x_1 - x_1(3x_1^2+3x_2^2+3x_3^2+x_4^2) - 2x_3x_4x_2 + k = 0$$

$$x_3' + (\mu - \xi \sin(\psi))x_3 - (\sigma + \xi \cos(\psi))x_4 + x_4(x_1^2+3x_2^2+3x_3^2+3x_4^2) + 2x_1x_2x_3 = 0$$

$$x_4' + (\mu + \xi \sin(\psi))x_4 - (-\sigma + \xi \cos(\psi))x_3 - x_3(3x_1^2+x_2^2+3x_3^2+3x_4^2) - 2x_1x_2x_4 = 0$$

(2.54)

These equations may be written in the following symbolic form :

$$\frac{dx}{dt} + f(x\ ;\ \mu,\ \sigma,\ \xi,\ k,\ \psi) = 0. \qquad (2.55)$$

The aim of the further analysis is to study the solutions of,

$$\frac{dx}{dt} + f_{\lambda_0}(x) = 0 \qquad (2.56)$$

as they vary with perturbations in the perameters.

Chapter 3

COMPUTATIONAL PROCEDURES

The equations determining the motion of the string are given in terms of the variables of order ϵ, and large time scale. The equations for the case of stationary solutions can be considered as a map :

$$f (x, \lambda) = v \qquad (3.01)$$

The solutions for the problem are given by intersection of this map with $v = 0$. Bifurcation points are special points in the solution domain, at which determinant of the Jacobian vanishes and the solution curve in (x, λ) in the neighborhood of these points are determined by the non-linear terms present in the map. From these points more than one branch of solution curves emerge, along each of which Jacobian is invertible. Obtaining these points is essential to plot the bifurcation diagram, and thereby study the steady state response of the system.

At the bifurcation points, the Jacobian is not invertible, which is same as the claim that one or more of the eigenvalues of Jacobian matrix are zero. The eigenvalues are the roots of its characteristic polynomial. The conditions under which the system of equations has a certain number of its eigenvalues zero, are specifed by the coefficients of the characteristic polynomial :

$$P(\alpha) = \alpha^4 + a_3\alpha^3 + a_2\alpha^2 + a_1\alpha + a_0 \qquad (3.02)$$

Hence, for example, to get a bifurcation point having two of its eigenvalues equal to zero, the set of equation to be solved is given as,

$$\begin{aligned} &\underline{f}(\, x, \lambda \,) = 0 \\ &a_o(\, x, \lambda \,) = 0 \\ &a_1(\, x, \lambda \,) = 0 \end{aligned} \qquad (3.03)$$

The set of equation so contructed has m>4 equalities to be satisfied. For this reason, the set of variables is augmented by allowing m-4 parameters to vary. Thus, given a guess value (x_o, α_o, β_o), when (x, α) are allowed to vary ,we try to obtain $(x_o', \alpha_o', \beta_o)$ for which the equations are satisfied.

3.1 *GAUSS NEWTON'S METHOD*

The set of equations (for example 3.03) to be solved can be represented symbolically as,

$$F(\, Y \,) = 0 \qquad \text{where, } Y = (\, x, \alpha)$$

$$F(\, Y_{k+1} \,) = F\,(\, Y_k \,) + F'(\, Y_k \,)\, \Delta Y_k + \cdots \cdots$$

$$F'^{T}(\, Y_k \,)\, F\,(\, Y_{k+1}) = F'^{T}(\, Y_k \,)\, F(\, Y_k \,) + F'^{T}(\, Y_k \,)\, F'(\, Y_k \,)\, \Delta Y_k$$

Taking, $F(\, Y_{k+1} \,) = 0$,

$$F'^{T}(\, Y_k \,)\, F(\, Y_k \,) = -F'^{T}(\, Y_k \,) F'(\, Y_k \,)\, \Delta Y_k \qquad (3.04)$$

The procedure runs as follows: for a given value of Y_k, ΔY_k, is obtained from the equation (3.04) using gauss-elemination technique. ΔY_k is the direction in which the value of the function

approaches zero.

$$Y_{k+1} = Y_k + \nu \Delta Y_k \qquad (3.05)$$

The value of ν is chosen so as to minimise the function in that direction. For this value of ν , Y_{k+1}, is obtained which is taken as guess value for the next iteration.

The termination condititon for the method may be taken as step-size (ν) being less than an error parameter, or the norm of the function being less than a given value.

This procedure constitutes the routine GAUSS_NEWTON. The routine is used to find a solution to the given set of equations in the programs BIFURCATION, LYAPUNOV and DIRECTION.

3.2 *METHOD FOR CALCULATION OF DERIVATIVE OF AUGMENTED EQUATION*

To calculate the bifurcation points, the function and their derivatives are to be evaluated repeatedly. To this end, the set of equations (3.01) are stored as a combination of array and records. The smallest entity that is to be dealt with is a term. In program BIFURCATION, which calculates the bifurcation point, the data structure that is used to store the information about a term is called TERM_1.

It has three fields

: COEFF stores the coefficient of the term.

: PWR stores coded information about the exponents of all variables and parameters.

: FUNC contains information about the symbolic

functions, such as Sin and Cos, present in the term. A boolean code TRNSD is used to check for corresponding variables.

An array of such records, having elements corresponding to the terms in the function is used to store functions in the program. Such a vector array is called FNX_1. On similar lines, vector functions are defined as a 1-dimensional array of FNV_1. 2-dimesional and 3-dimensional arrays of functions are similarly defined.

Operations of differentiation (DIFFERENTIATE) and evaluation (EVAL) neccesary for computation of bifurcation point are coded so as, to be performed on such structures. These operations make it possible to work with function as basic objects.

3.3 *OPERATIONS ON COEFFICIENTS OF CHARACTERISTIC POLYNOMIAL*

For evaluation and differentiation of the coefficients of characteristic polynomials, it is neccesary to introduce another data structure. It is termed ELEMENT_CD in the program. It is an array of another record called COEFF_S, each of which denotes the coefficients of the characteristic polynomial ($P(\alpha)$). These are inputed as functions of entries of an n x n matrix (four at this stage). COEFF_S are functions with each term an n-tuple at maximum, denoting the combination of entries to be multiplied and subsequently added to give values for the coefficients. Thus operation of evaluation (EVAL_COEFF_S) and differentiation (EVAL_DIFF_COEFF) on such structure may be easily defined.

The coefficients are differentiated in the following way.

$$\frac{\partial}{\partial x_j} a_{\iota} = \sum_{\iota,m} \frac{\partial}{\partial A_{\iota,m}} a_{i} \quad \frac{\partial}{\partial x_j} A_{\iota,m} \qquad (\text{where, } 1 \leq \iota,m \leq 4) \qquad (3.06)$$

The symbol $A_{\iota,m}$ denotes the entry of the matrix. The first term under summation in formula (3.06) is obtainable using the code of ELEMENT_CD. The second term can be computed before hand and stored in three dimensional array of reals.

3.4 *CALCULATION OF PROJECTION MATRIX*

The procedures mentioned previos sections give the bifurcation points having at least the given number of null space dimensions. The next step consists of projecting the map such that variables are partitioned to span the null space and its compliment. To this end, first the vectors spanning the null space of Jacobian matrix at bifurcation point, are computed. The compliment vectors are taken to be orthogonal to them. The vectors are arranged to form a projection matrix in such a way so that, if the null space is m-dimensional, variables $x_1, \cdots, x_{4-m}$ denote the compliment and rest span the null space of the mapping.

This part of operation is performed in the program PROJ_MAT. Given a singular Jacobian matrix by the program BIFURCATION, PROJ_MAT computes the required projection matrix.

3.5 *EXPANSION OF THE MAPPING ABOUT THE BIFURCATION POINTS*

The next step in the proposed analysis consists of expanding the function about given bifurcation point. This part of the

operations are performed in the program LYAPUNOV. The given function is a 4-dimensional mapping with 5-parameters and 4-variables. For purpose of expansion no distinction is made between the parameters and the variables. The process of expansion is performed by translating the domain coordinates.

The description of the basic entity, term of a function, in this program, requires only two fields,

: C stores coefficient of the term
: P contains coded information about the powers to which each variable is raised.

Routines are written to perform various operations on the functions which are coded using such structures.

: ADD_TERM adds a term to a given function.
:MLTP_TERM multiplies two term and returns the resulting term.
: ADD_FN adds a function to a given function.
: SCALAR_FN multiplies a function by a scalar.
: SORT_FN sorts a given function to resemble a Taylor series.
: MLTP_FN multiplies two functions and returns the resulting one.
: COMPACT removes any term having coefficient less than a given value (EPS) from between any two terms.
: EXPAND_FN translates the origin of the variable coordinates by the given position vector.
: SUBST_FN substitutes a given function in place of a variable in a given function.

The program LYAPUNOV using above mentioned procedures expands

the equations about the bifurcation point :

$$v = f(x, \lambda)$$

$$= f(\tilde{x} + x_{\beta}, \tilde{\lambda} + \lambda_{\beta})$$

where, $(x_{\beta}, \lambda_{\beta})$ denote the bifurcation point, hence

$$v = g(\tilde{x}, \tilde{\lambda}) \tag{3.07}$$

This expansion process makes purely numerical valued terms in the $g(\tilde{x},\tilde{\lambda})$ to be equal to zero, but the Jacobian is same as that without expansion. Next step consists of rotation of the variable coordinate system using the projection matrix already computed by the program PROJ_MAT.

$$\tilde{x} = P\ \bar{x}$$
$$\bar{v} = P^{-1}\ g(P\ \bar{x}, \tilde{\lambda}) \tag{3.08}$$

Let $\bar{x}_{\beta}$ denote the variables spanning the null space and $\bar{x}_{\alpha}$ spanning the compliment of the null space. Similarly, let $\bar{v}_{\alpha}$ be the variable spanning the range and $\bar{v}_{\beta}$ the variable spanning the compliment of range.

$$\bar{v}_{\alpha} = g_{\alpha}(\bar{x}_{\alpha}, \bar{x}_{\beta}, \lambda)$$
$$\bar{v}_{\beta} = g_{\beta}(\bar{x}_{\alpha}, \bar{x}_{\beta}, \lambda) \tag{3.09}$$

As $D_{\bar{x}_{\alpha}} g_{\alpha}(\bar{x}_{\alpha}, \bar{x}_{\beta}, 0)$ is invertible, by inverse function theorem $\bar{x}_{\alpha}$ can be expressed locally in terms of $(\bar{x}_{\beta}, \bar{\lambda})$ for $v = 0$. In other words,

$$0 = g_{\alpha}(\bar{x}_{\alpha_o}, \bar{x}_{\beta}, \bar{\lambda}) \tag{3.10}$$

can be solved for $\bar{x}_{\alpha_o}$. Multiplying equation (3.10) with,

$$\left[D_{x_\alpha} g_\alpha(x_\alpha , x_\beta , 0)\right]^{-1}$$

we have,

$$0 = x_{\alpha_o} - G(x_{\alpha_o} , x_\beta , \lambda) \tag{3.11}$$

For $\| x_{\alpha_o} \|$, $\| x_\beta \|$ and $\| \lambda \|$ to be much less than one, following procedure helps in finding an expression for x_{α_o} in terms of (x_β , λ) valid for small values.

$$\begin{aligned} x_{\alpha_o}^{k+1} &= G(x_{\alpha_o}^{k} , x_\beta , \lambda) \\ x_{\alpha_o}^{1} &= 0 \end{aligned} \tag{3.12}$$

The above procedure converges, as

$$\begin{aligned} &\| x_\alpha^{k+1} - x_\alpha^{k} \| \leq \| D_{x_\alpha}^{k} G(x_\alpha^{k} , x_\beta , \lambda) \| \| x_\alpha^{k} - x_\alpha^{k-1} \| \\ &\| D_{x_\alpha} G(x_\alpha , x_\beta , \lambda) \| = O(|x_\alpha| , |x_\beta| , |\lambda|) \ll 1 \end{aligned} \tag{3.13}$$

These operations are done by the routine SOLVE in program LYAPUNOV aided by procedures FIRST_APPROX and PROCESS.

Having obtained solution,

$$x_{\alpha_o} = x_{\alpha_o} (x_\beta , \lambda) \tag{3.14}$$

the solution is substituted in the equation (3.09.ii) the map to be studied is reduced to,

$$v_\beta = F (x_\beta , \lambda) \tag{3.15}$$

In these equations , $D_{x_\beta} F(x_\beta , 0)$ is nilpotent.

3.6 *PROCEDURE TO OBTAIN TANGENT VECTORS TO THE SOLUTION CURVES AT THE BIFURCATION POINTS*

Following procedure is used to compute the tangent vectors at the bifurcation points. At bifurcation points the main problem is to find all the stationary solution curves around that point. This procedure is used to find tangent directions to the curves unfolding from that point. This stage of computations is performed in the program DIRECTION.

Let Y denote (x_β, α) where α is the parameter with which the bifurcation diagram is to be sketched and F (Y) is the reduced map ,*i.e.* restriction to the null space. The directions are obtained in the process of solving the set of equations given below.

$$
\begin{aligned}
0 &= F\ (\ Y\ ) \\
0 &= [\ D_y F(\ Y\ )\ ]\ Z \\
0 &= |\ Y\ |^2 - \xi^2 \\
0 &= \xi^2 \\
0 &= |\ Z\ |^2 - 1
\end{aligned}
\qquad (3.16)
$$

Given a vector function the procedure MAKE_FUNC assembles the above set of equations.

If Y denotes n unknowns, the above set of equation are in terms of 2n+1 unknowns. These equations are solved wih Gauss-Newton method presented earlier. The solutions as apparent from formulation of the equation set, has all its components zero except those denoted by Z. It can be claimed that Z represents one of the

many possible null directions.

Different possible directions are obtained for different set of guess values for the solution. ξ is generally initialised with value of order one. Y and Z ,for initial guess are taken to be equal and of order one.

Having obtained all the possible tangent direction a small step is taken in each direction, and all the variables in the compliment of the kernel are also evaluated using equations (3.14).

This sets of (x_α , x_β , α) are used as reliable guesses for each of the possible solution emanating from the bifurcation point.

3.7 *CONTINUATION PROCEDURE*

The above procedure gives good guesses for one point on all possible solution curves around the given bifurcation point. The aim of the procedures in this section is to trace the variables as they vary with a parameter α. These procedures constitute the CONTINUATION program. The information available are the bifurcation point and a guess point near a solution curve , relative to bifurcation point.

The technique used to construct a solution curve consists of first obtaining tangent at a given point in (x, α) on the solution curve and moving in that direction by a step-size. The value of step-size is governed by various considerations. The most important of them is that while moving along the tangent direction, the norm of the function does not increase beyond a preset value. Constraints are put directly on the step size, as well. Its

maximum value is set to bring a homogeneity in the distance between two solutions on the same curve. When the value of step size goes below a certain value the procedure is coded to stop and plotting of the next solution starts from the given guess point near the bifurcation point.

Taking step in following fashion, provides a good guess values for the next stage of computation.

Let, $Y = (x, \alpha)$,

Y_o be the point on solution curve known before hand.
δY_o be the tangent at the point,

$$Y_1 = Y_o + \text{step} \times \delta Y_o . \tag{3.17}$$

This process is repeated every time a guess for new solution point is required, given the solution at which the Jacobian is invertible. At bifurcation points, results of the program DIRECTION are used as a guess for solution in neighborhood of the singular point.

The value Y_1 so obtained, is used as guess value to solve the following augmented equations by Gauss-Newton 's method :

$$\begin{aligned} 0 &= F (Y) \\ 0 &= \| Y-Y_o \| - \text{step}^2 \end{aligned} \tag{3.18}$$

This procedure returns solution as Y_1'. The tangent vector at this point ,$\delta Y_1'$, is found by solving following system of equation.

$$0 = [D_y F(Y_1')] \delta Y_1' \tag{3.19}$$

A further check is performed on $\delta Y_1'$ so as not to take step in direction opposite to that taken in previous cycle. Hence,

if,

$$\delta Y_1' \cdot \delta Y_0 < 0 \qquad (3.20)$$

then the sign of $\delta Y_1'$ is changed. Y_1' is now taken as Y_0 and $\delta Y_1'$ as δY_0 for the next iteration. The curve length is incremented by step size.

3.8 *DYNAMIC STABILITY*

The dynamic stability at each of these points is predicted by Routh-Hurwitz criteria. The criteria predict the sign of the real part of eigenvalues by merely checking some conditions on the coefficients in the characteristic polynomial. For the conditions defined by,

$$\begin{aligned} & a_3 > 0 \\ & a_3 a_2 - a_1 > 0 \\ & a_3 a_2 a_1 - a_1^2 - a_3^2 a_0 > 0 \\ & a_3 a_2 a_1 a_0 - a_1^2 a_0 - a_3^2 a_0^2 > 0 \end{aligned} \qquad (3.21)$$

the real part of all the eigen values are negative.

In the program **CONTINUATION** , record structure is created to store the conditions as functions of coefficients of characteristic polynomial. As the neccesary data structure for evaluation of coefficients are already available, the computations for stability can easily be carried out. The Jacobian of the function at that point is passed to the boolean function **STABLE** . It computes the coefficients of the characteristic polynomial for the given matrix. The conditions coded in **DYN_ST** are verified for the given values of the coefficients. If the conditions are satisfied then the function assumes value *true*.

CHAPTER 4

LOCAL ANALYSIS OF BIFURCATION CHARACTERISTICS

The formulation and analysis of equation of motion for string, done in the previous chapters has been based on the local study of the equations about certain critical points, for which non-linear nature of the problem is prominent. The governing equations have been derived considering the geometeric relationship between strain and displacement to be a perturbation from that for the linear case. The general motive for application of perturbation method to the problem is to facilitate the analysis of the problem in the neighborhood of resonance frequency of the linearised system, as in such a situation linear theory fails to give realistic explanation for the system response.

Local study of bifurcation characteristics shown by singular stationary solutions is an extension of same ideas. Such an analysis is imperative for prediction and understanding of the general nature of the response. The phenomenon of bifurcation is intricately related to the singular points of the map describing the general non-linear behavior.

Bifurcation phenomena occur as a result of interactions between solutions predicted by the model. Such an interaction, for example, may be an exchange of stability between the solutions. As a result of such exchange the response characteristics of the system change drastically. In case of strings, there exist conditions under which for slight variation in the parameters leads to planar motion giving way to non-planar whirling.

4.1 *DYNAMIC STABILITY AT REGULAR POINTS*

Stability of the system for a given value of stationary solution response can be predicted by the analysis of the Taylor series expansion about that response value. Let,

$$\frac{dx}{dt} = f_\lambda(x) \qquad (4.01)$$

$$0 = f_\lambda(x_o).$$

Then with a change in the coordinates,

$$y = x - x_o$$

the differential equation (4.01) may be written as,

$$\frac{dy}{dt} = f_\lambda(y) \qquad (4.02)$$

$$= A_\lambda y + g_\lambda(y)$$

where A_λ is an n x n matrix. This matrix can be reduced to Jordan canonical form using a transformation,

$$z = Py$$

$$\frac{dz}{dt} = P\,A_\lambda\,P^{-1}z + P\,g_\lambda(P^{-1}z) \qquad (4.03)$$

$$\frac{dz}{dt} = B_\lambda z + G_\lambda(z)$$

where P is taken to be such that det(P) = 1.

About regular points the Jacobian matrix A_λ has non-zero, distinct eigenvalues though they may be complex, hence

$$\frac{d}{dt}z_i = \alpha_i z_i \;, \qquad i = 1, 2, \ldots\ldots, n. \qquad (4.04)$$

In equation (4.04) higher order terms have been neglected. It is clear that in such cases stability can be determined by computing the eigenvalues of the matrix A_λ. If all the eigenvalues have their real parts as negative, then each of the z_i have a form showing

exponential decay with time. For such cases the solutions $z_i = 0$ (or $x = x_o$), are said to be stable.

Routh - Hurwitz criteria provide an easier procedure to check the signs of the real parts of eigenvalues of a given matrix by just checking for certain conditions on coefficients of the characteristic polynomial (Chapter 3).

A stable solution becomes unstable when at least one of the eigenvalues crosses the imaginary axis in the complex plane depicting them, from negative real axis side to positive one. If they cross for non-zero imaginary value, then it can be argued that there are even number of such eigenvalues and the system now becomes capable of showing an oscillatory motion in the respective eigen directions. This is a more complex phenomenon than that in which the eigenvalues are real, and is refered to as Hopf bifurcation.

This work is an attempt to analyse the cases in which the eigenvalues cross imaginary axis at the origin. For these cases local analysis of the problem about the singular point assumes more importance, as negelecting the occurance of Hopf bifurcation, the change in stability on a particular branch of solutions which have invertible Jacobian, is possible only when a singular point is encountered. Thus, branches in the bifurcation diagram on which the Jacobian is invertible are capable of showing no bifurcation behavior other than Hopf bifurcation.

4.2 *STABILITY AT A SINGULAR POINT*

As, apart from occurance of Hopf bifurcation, all stability related phenomena occur only at singular points, and stability at a branch of solution with invertible Jacobian, is retained as

determined at the singular point from which the branch originated, a local stability analysis of each of such branches at singular points provides major clues about the general behavior of the system. The motive, hence, is to determine the stability at each of such curves that emanate from a singular point, locally.

4.3 *ORGANISATION OF SOLUTIONS ABOUT THE SINGULAR POINTS*

The singular points form organising centres for the solutions. The conditions under which a singular point exists can be described by constraints on the space of parameters. This is quite apparent from the fact that for any random value of λ the solution has invertible Jacobian. The order of a singularity may be described by the number of equations that need to be satisfied by the parameters for its existence .

These equations define a smooth manifold in the space of parameters. Corresponding to these values of parameters, the Taylor expansions of the map are elements of the same subset in the set of Taylor series of all smooth functions.

Manifolds so defined may intersect to give singularities of higher order. The singular points can themselves be classified, on such considerations. Thus, it can be argued that in a map $f_\lambda(x) \to v$, there exist maximal singularities for some values of the parameters, as x is an element of $\mathbb{N}$, a smooth manifold. In the case of equation of motion of string,

$$f(\lambda, x) = v$$
$$v = \frac{dx}{dt} \qquad (4.05)$$

where v = 0 fo stationary solutions. For x = 0 and λ = 0, the singularity is maximal : it has 4-dimensional null space.

Organisation of singularities of given map resembles a tree-structure in which easing of certain restrictions or variation of parameters in a certain direction gives rise to singularity of lesser order. This process can be continued upto the singular solution of zeroth order, *i.e*, invertible solutions. The analysis about singular point of highest order, provides greater understanding about the organisation of solutions and a clearer picture of the phenomena of bifurcation that the system might show.

4.4 *LOCAL STUDY OF SINGULARITIES*

A general problem faced in the study of non-linear problems is the presence of multiple solutions in the neighborhood of singular points and ways of obtaining all of them. However, strict non-linear nature of the map in the neighborhood of these points provides a methodology to study such problems by restricting the analysis to broader implications.

As linear changes in the domain and range do not affect the zero set of the Taylor series expansions in terms of bifurcation characteristics, equivalence classes may be defined on the set of all smooth mappings, under an equivalence relation based on such characteristics. The maps which have same bifurcation characteristics are members of the same equivalence class.

For such an analysis, the object of study becomes the restriction of the function to a small neighborhood of the singular points:

$$\begin{aligned} f &: (\mathbb{N} \cap \mathbb{U}) \longrightarrow \mathbb{P} \\ \mathbb{N} &\subseteq \mathbb{R}^n \\ \mathbb{P} &\subseteq \mathbb{R}^p \end{aligned} \tag{4.06}$$

Where $\mathbb{U}$ is a small open neighborhood of the singular points.

The technique proposed for analysis of bifurcation behavior relies on the fact that, given an expansion about a singular point, a simpler approximate map can be obtained which, in the space of all smooth maps, is an element of the same equivalence class, defined by the equivalence of bifurcation properties.

The bifurcation related analysis can then be carried out on such an equivalent map. Then, a correspondence can be found between the bifurcation behavior and the variation of parameters.

The set of maps that show similar bifurcation behavior in vicinity of a singular point can be identified with algebraic concept of *germs*.

4.4.1 *Germs of smooth mappings*

Given a point $x \in \mathbb{N}$, consider the set of all smooth mappings from domain $\mathbb{U}$, which is a neighborhood of x in $\mathbb{N}$, to $\mathbb{P} \subseteq \mathbb{R}^p$. On this set an equivalence relation, $\sim$, may be introduced .

Given two such mappings $f_1: \mathbb{U}_1 \longrightarrow \mathbb{P}$ and $f_2: \mathbb{U}_2 \longrightarrow \mathbb{P}$, $f_1 \sim f_2$, when there exists a neighborhood $\mathbb{U}$ of x in $\mathbb{N}$ depending on f_1 and f_2 for which the resrtiction $f_1|_{\mathbb{U}}$ and $f_2|_{\mathbb{U}}$ coincide. The equivalence classes under this relation are called smooth germs of mappings $\mathbb{N} \longrightarrow \mathbb{P}$ at x. The elements of same equivalence class are called representatives of same germ.

Thus, it can be claimed that under the equivalence relation defined by type of bifurcation behavior, *i.e.* , equivalence under linear transformations, representatives of the same germ show same bifurcation charateristics.

If f_1 and f_2 are representatives of the same germ, then $f_1(x) = f_2(x)$. In view of this fact, notation adopted to denote the

germ is, $f : \mathcal{E}(\mathbb{N}, x) \longrightarrow \mathcal{E}(\mathbb{P}, y)$, where x and y are called source and target of the germ respectively. Given two germs,

$$f : \mathcal{E}(\mathbb{N}, x) \longrightarrow \mathcal{E}(\mathbb{P}, y) \quad \text{and,}$$
$$g : \mathcal{E}(\mathbb{P}, y) \longrightarrow \mathcal{E}(\mathbb{Q}, z),$$

operation of composition may be defined as,

$$f \circ g : \mathcal{E}(\mathbb{N}, x) \longrightarrow \mathcal{E}(\mathbb{Q}, z) \tag{4.07}$$

This definition does not depend on a particular choice of representatives. Under operation of composition there may exist an inverse if,

$$f \circ g = g \circ f = I \tag{4.08}$$

Inverse mapping theorem applied to germs means that a germ is invertible if and only if its differential is invertible. The Rank of a germ is defined to be that of its differential or Jacobian.

4.4.2 *The a - equivalence of germs*

Two germs f_1 and f_2 are said to be a-equivanlent when there exist invertible germs g and h for which, if

$$f_1 : \mathcal{E}(\mathbb{N}_1, x_1) \longrightarrow \mathcal{E}(\mathbb{P}_1, y_1)$$
$$f_2 : \mathcal{E}(\mathbb{N}_2, x_2) \longrightarrow \mathcal{E}(\mathbb{P}_2, y_2)$$
$$g : \mathcal{E}(\mathbb{N}_2, x_2) \longrightarrow \mathcal{E}(\mathbb{N}_1, x_1)$$
$$h : \mathcal{E}(\mathbb{P}_2, y_2) \longrightarrow \mathcal{E}(\mathbb{P}_1, y_1)$$

then,

$$f_1 \circ g = h \circ f_2 \tag{4.09}$$

The basic objects of study are germs $f : \mathcal{E}(\mathbb{N}, x) \longrightarrow \mathcal{E}(\mathbb{P}, y)$. Hence, this is equivalent to $f : \mathcal{E}(\mathbb{R}^n, 0) \longrightarrow \mathcal{E}(\mathbb{R}^p, 0)$. The set of all such germs is termed $\mathcal{E}^0_{n,p}$.

$\mathcal{E}^{0}_{n,p}$ has the structure of a vector space induced from that or $\mathbb{R}^p$. Denote $\mathcal{R}_n$ to be the set of invertible germs $(\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$, which is a group under operation of composition. $A_{n,p}$ denotes the poduct group $\mathcal{R}_n \times \mathcal{R}_p$. The action of the group $A_{n,p}$ on $\mathcal{E}^{0}_{n,p}$ is given as,

$$(g , h) \circ f = h \circ f \circ g^{-1} \qquad (4.10)$$

Given such an action , a-equivalence of x and y, $\in \mathcal{E}^{0}_{n,p}$ implies that $y = g \circ x$ for some $g \in A_{n,p}$ and the zero set of each encompasses same non-linear behavior. Thus, if x and y are a-equivalent, then they are so through a pair of invertible germs (h, g) which is an element of the product group $A_{n,p}$. The equivalence classes under such an action are called orbits. The orbit through x is given by definition as the equivalence class which contains x.

$$A_{n,p} \circ x = \{ g \circ x : g \in A_{n,p} \} \qquad (4.11)$$

The problem of studying the types of bifurcation of the zero set of the mapping (4.05) is identical to that of studying the orbits under this action.

4.4.3 *Unfolding of germs*

An r-parameter unfolding of $f_o : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^q, 0)$ is the germ $F : (\mathbb{R}^r \times \mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^r \times \mathbb{R}^m, 0)$ given by the formula :

$$F(u, x) = (u , f(u, x)) \qquad (4.12)$$

4.4.4 *Deformation of germs*

An r-parameter deformation of $f : (\mathbb{R}^m ,0) \longrightarrow (\mathbb{R}^q, 0)$ is a germ $f : (\mathbb{R}^r \times \mathbb{R}^m , 0) \longrightarrow (\mathbb{R}^q , 0)$ with $f(0,x) = f_o(x)$.

THEOREM 4.1 *([1], Section 5.4.1)*

Let $G : \mathcal{E}\,\mathbb{R}^n, 0 \rangle \longrightarrow \mathcal{E}\,\mathbb{R}^p, 0 \rangle$ be a germ of rank r, the there exists an invertible germ $h : \mathcal{E}\,\mathbb{R}^n, 0 \rangle \longrightarrow \mathcal{E}\,\mathbb{R}^n, 0 \rangle$ fo which, $F = G \circ h$ is an r-parameter unfolding of a germ of rank 0.

Thus, if the map has γ-dimensional null space then it is a-equivalent to another map which is $(p-\gamma)$-paramter unfolding of a germ of rank zero.

In Chapter 3, methods have been discussed to restrict the equations to the null space .For equation of motion of strings, if the null space is γ-dimensional, then forms denoting direct parameterisations for (4 - γ) solution variables in the compliment of the kernel, can be obtained from equation (3.14).

$$x_{\alpha_o} = x_{\alpha_o}(x_\beta, \lambda) \tag{4.13}$$

The Taylor expansion of governing equations about a given singular point may be partitioned into kernel and its compliment, and by making a change in the variables,

$$\begin{aligned} y_\alpha &= x_\alpha - x_{\alpha_o}(x_\beta, \lambda) \\ y_\beta &= x_\beta , \end{aligned} \tag{4.14}$$

the set of equations are a-equivalent to following map :

$$\begin{aligned} u_\alpha &= y_\alpha \\ u_\beta &= g(y_\alpha, y_\beta, \lambda). \end{aligned} \tag{4.15}$$

The aim of further theory is to reduce the dimensionality of the unfolding of $g(y_\beta, 0, 0)$.

4.4.5 *The a-equivalence of r-parameter unfoldings*

Two r-parameter unfolding F_1, F_2 of a germ $f: (\mathbb{R}^m, 0) \to (\mathbb{R}^q, 0)$ are said to be a-equivalent when there exist r-parameter unfoldings I_m, I_q of the germs at zero of identity maps on $\mathbb{R}^m$, $\mathbb{R}^q$ respectively for which

$$\begin{aligned} F_1 &: (\mathbb{R}^r \times \mathbb{R}^m, 0) \to (\mathbb{R}^r \times \mathbb{R}^q, 0) \\ F_2 &: (\mathbb{R}^r \times \mathbb{R}^m, 0) \to (\mathbb{R}^r \times \mathbb{R}^q, 0) \\ I_m &: (\mathbb{R}^r \times \mathbb{R}^m, 0) \to (\mathbb{R}^r \times \mathbb{R}^m, 0) \\ I_q &: (\mathbb{R}^r \times \mathbb{R}^q, 0) \to (\mathbb{R}^r \times \mathbb{R}^q, 0) \end{aligned}$$

$$F_1 \circ I_m = I_q \circ F_2 \qquad (4.16)$$

The pair (I_m, I_q) are refered to as a-equivalence between F_1 and F_2.

An r-parameter unfolding F of f is a-trivial when it is a-equivalent to the constant r-parameter unfolding $(\mathbb{R}^r \times \mathbb{R}^m, 0) \to (\mathbb{R}^r \times \mathbb{R}^m, 0)$ of f given by $(u, x) \to (u, f(x))$. f is called a-stable when every unfolding of f is trivial.

4.4.6 *Contact equivalence*

Taking graphs of representatives $f_1: (\mathbb{N}_1, x_1) \to (\mathbb{P}_1, y_1)$ and $f_2: (\mathbb{N}_2, x_2) \to (\mathbb{P}_2, y_2)$, which will be smooth manifolds of $\mathbb{N}_1 \times \mathbb{P}_1$ and $\mathbb{N}_2 \times \mathbb{P}_2$ through points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, contact equivalence is defined to be the pair (h, H) of invertible germs for which one has following commutative diagram

$$\begin{array}{ccccc} (\mathbb{N}_1, x_1) & \xrightarrow{\mathcal{G}_1} & (\mathbb{N}_1 \times \mathbb{P}_1, z_1) & \xrightarrow{\Pi_1} & (\mathbb{N}_1, x_1) \\ \downarrow h & & \downarrow H & & \downarrow h \\ (\mathbb{N}_2, x_2) & \xrightarrow{\mathcal{G}_2} & (\mathbb{N}_2 \times \mathbb{P}_2, z_2) & \xrightarrow{\Pi_2} & (\mathbb{N}_2, x_2) \end{array}$$

where $\mathcal{S}$ is the inclusion mapping and, Π is the projection mapping.Hence, H is given by

$$H(x,y) = (h(x) , \theta(x,y))$$
$$\theta(x,y_1) = y_2 \qquad (4.17)$$

Thus, f_1 and f_2 are contact equivalent or k-equivalent when there exists a contact equivalence (h, H) for which

$$H \circ (1,f_1) = (1,f_2) \circ h. \qquad (4.18)$$

This is shown schematically in the diagrams given below.

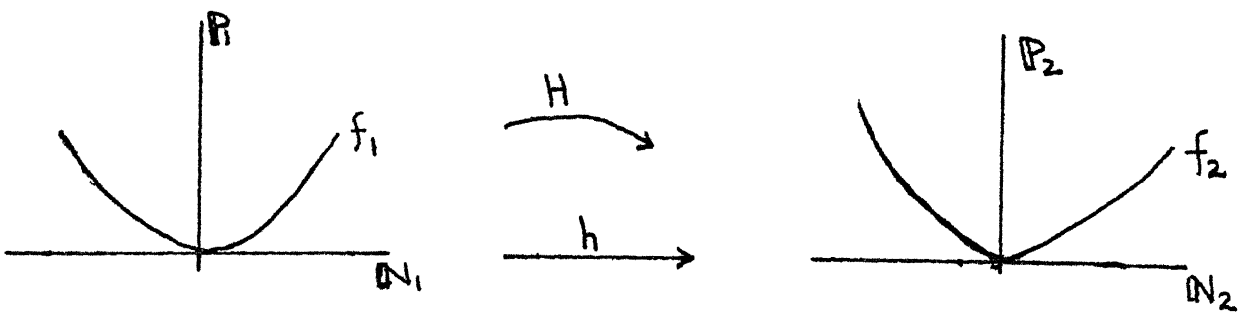

Fig 2

Thus, two germs $(\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ lie on the same orbit under this action if and only if they are k-equivalent. Thus, the problem of classifying germs under k-equivalence is that of classifying the orbits under above action, the k-orbits in $\mathcal{E}^0_{n,p}$.

THEOREM 4.2 *([1] , Section 5.4.2)*

Let F, $F' : (\mathbb{R}^r \times \mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^r \times \mathbb{R}^q, 0)$ be r-parameter unfoldings of f_F , $f_{F'}$, of rank 0; If F and F' are a-equivalent then f_F , $f_{F'}$, are k-equivalent.

4.4.7 *Morphism*

Let F , G be r and s parameter deformations of f. A morphism from F to G is a pair (H, K) , with K a k-equivalence of r parameter deformations , and H a change of parameter, for which F is k-equivalent to induced deformation $G \circ H^{-1}$ under K.

4.4.8 *Versal deformation*

A deformation G of f is k-versal when for any deformation F there exist a morphism from F to G . A minimal parameter k-versal deformation is called k-universal.

The assignation $F \longrightarrow f_F$ induces a mapping from a-equivalence classes of germs to k-equivalence classes. Given an s-parameter deformation $f : (\mathbb{R}^s \times \mathbb{R}^m , 0) \longrightarrow (\mathbb{R}^q , 0)$ which is submersive, the zero set $v_f = f^{-1}(0)$ will be a smooth submanifold of $\mathbb{R}^s \times \mathbb{R}^m$ of codimension q , i.e., a hypersurface defined by q-equations in a space of s+m dimensions. We define the restriction $\Pi_f : (v_f, 0) \longrightarrow (\mathbb{R}^s, 0)$ to be the germ at 0 of restriction to v_f of the projection germ $\Pi : \mathbb{R}^s \times \mathbb{R}^m \longrightarrow \mathbb{R}^s$.

THEOREM 4.3 *([1] , Section 5.4.3)*

Let $f, g : (\mathbb{R}^s \times \mathbb{R}^m , 0) \longrightarrow (\mathbb{R}^q , 0)$ be k-versal s-parameter deformation of germ f_o, $g_o : (\mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^q, 0)$ of rank 0 ; if f_o, g_o are k-equivalent then Π_f , Π_g are a-equivalent.

Let $F : (\mathbb{R}^r \times \mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^r \times \mathbb{R}^q, 0)$ be an r-parameter unfolding of a germ f_F of rank 0, given by $F(u, x) = (u, f(u, x))$. To F, we associate a germ $D_F : (\mathbb{R}^r \times \mathbb{R}^q \times \mathbb{R}^m , 0) \longrightarrow (\mathbb{R}^q , 0)$, given by $(u, w, x) \longrightarrow -w + f(u, x)$. Thus, D_F is an (r+q) parameter submersive deformation . The geometric connection between F and D_F is that v_{D_F} is just graph (f), and Π_{D_F} can be identified with f.

THEOREM 4.4 *([1] , Section 5.4.4)*

F is a a-stable germ if and only if D_F is a k-versal deformation of f_F.

THEOREM 4.5 *([1] , Section 5.4.5)*

Let $F, G : (\mathbb{R}^r \times \mathbb{R}^m, 0) \longrightarrow (\mathbb{R}^r \times \mathbb{R}^q, 0)$ be stable r-parameter unfoldings of the germs f_F and f_G of rank 0; if f_F and f_G are k-equivalent, then F, G are a-equivalent.

4.5 *JETS OF SMOOTH MAPPINGS*

The jet-space $J^k(n, p)$ means the real vector space of all mappings $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ each of whose component is a polynomial of degree $\leq k$ in the standard coordinates $(x_1, \cdots, x_n)$ in $\mathbb{R}^n$ with zero constant term. The k-jet of a germ ($J^k f$) corresponds to the finite initial segment (degree $\leq$ k) of the Taylor series about source and target.

In the set $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^p)$ given a smooth mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, positive numbers ϵ and R, and an integer $k \geq 0$, a *fundamental neighborhood* to f comprises of smooth mappings $g : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ for which , for all $x \in \mathbb{R}^n$ with $| x | \leq R$, one has

$$\| J^k f(x) - J^k g(x) \| \leq \epsilon \qquad (4.19)$$

with $\| \circ \|$ a fixed norm on the jet space $J^k(\mathbb{R}^n, \mathbb{R}^k)$.

A subset $x \subseteq \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^p)$ is called *dense*, when given any smooth mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, and any fundamental neighborhood (v) of f one can find smooth maps $g : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, in x with $g \in v$.

4.6 *FIRST ORDER SINGULARITY SETS*

A singular point of smooth mapping $f : \mathbb{N} \longrightarrow \mathbb{P}$ is a point $x \in \mathbb{N}$ for which the rank of the differential or Jacobian falls below its possible maximal value of min(n, p).

A natural way of distinguishing one singular point from another is by the actual value taken by the rank of the differential. To this end, first order singularity sets are defined as follows :

$$\Sigma^i f = \{ x \in \mathbb{N} : D_x f \text{ has kernel of dimension } i \} \qquad (4.20)$$

Σ^i is a smooth submanifold of $J^1(n,p)$ of codimension $i(p-n+i)$. This is the number of equations or conditions that can be imposed on a mapping to be an element of Σ^i.

4.7 *CLASSIFICATION OF SMOOTH MAPPINGS*

For maps $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ first order singularity sets $\Sigma^i f$ could be defined as discussed earlier . If these are smooth manifolds then, second order sets can be defined on the restrictions to the first order ones, $\Sigma^{i,j} = \Sigma^j (f \mid \Sigma^i f)$. This process can be continued. This way higher order *Thom 's singularity sets* of f may be obtained.

The problem of obtaining these submanifolds $\Sigma^{i_1,\ldots,i_k}$ in the jet-space $J^k(n,p)$,such that for generic $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$, they determine the k^{th} order singularity sets, was solved by *Boardman (1967)*.

4.7.1 *Real algebra*

Real algebra consists of structure of a real vector space V together with a bilinear mapping V x V $\longrightarrow$ V , written as (x, y) $\longrightarrow$ x.y and called algebra product. The real vector space $\in_n$ of all germs $f : (\mathbb{R}^n, x) \longrightarrow (\mathbb{R}, y)$ of smooth functions is endowed with algebra product induced from that on the reals.

An ideal is a subset of a ring closed under operations of addition and algebra product. The algebra product of an element of ideal with any element of ring is an element of ideal. In $\in_n$ this can be identified with the set of all germs in $\in_n$ with 0 as target.

4.7.2 *Boardman symbol (11, Section 5.5)*

Let I be a finitely generated ideal in the algebra $\in_n$, $f_1, \cdots, f_p$ be the generators for I, and $y_1, \cdots, y_n$ be the system of coordinates in $\in_n$. Given an integer $s \geq 1$ we define Δ_s to be the ideal I + I', where I' is the ideal generated by all s x s minors of the Jacobian matrix.

Given an ideal $I \subseteq \in_n$ following notation is used :

$$\Delta^s I = \Delta_{n-s+1} I$$

and $\Delta^1, \Delta^2, \cdots, \Delta^n$ are refered to as successive Jacobian extensions of the ideal. As a result of this definition following inclusions hold.

$$I = \Delta^0 I \subseteq \Delta^1 I \subseteq \Delta^2 I \subseteq \cdots \subseteq \Delta^n I \qquad (4.21)$$

I is said to be *proper*, when $I \neq \in_n$. the critical Jacobian extension of I is the last ideal $\Delta^{i_1} I$ in the sequence (4.21) which is proper. It in turn has a critical jacobian extension $\Delta^{i_2,i_1} I$. In this way an ascending sequence $\Delta^{i_1} I$, $\Delta^{i_2,i_1} I$, ... , can be obtained. I is said to have $(i_1, i_2, \ldots)$ as Boardman symbol.

Thus, the Boardman symbol of a germ $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ is defined by that of the ideal I_f generated by the components of $f_1, \cdots, f_p$.

THEOREM 4.6 *(11 , Section 5.5.2)*

The Boardman symbol of a germ $(\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ is contact invariant, i.e., if two such germs are k-equivalent then they have same Boardman symbol.

The first k integers in Boardman symbol of a germ depend only on the k-jet of f. A neccesary and sufficient condition for the set $\Sigma^{i_1,\ldots,i_k} \subseteq J^k(n, p)$ to be non-empty is that following conditions should be satisfied :

$$\begin{aligned} & n \geq i_1 \geq i_2 \cdots \cdots \geq 0 \\ & i_1 \geq n - p \\ & \text{if } i_1 = n - p \text{ then } i_1 = i_2 = \cdots \cdots = i_k \end{aligned} \tag{4.22}$$

THEOREM 4.7 *([1] , Section 5.5.5)*

If the k^{th} order singularity set $\Sigma^{i_1,\ldots,i_k} \subseteq J^k(n, p)$ is non-empty then it is a *smooth submanifold* of codimension given by the following formula.

$$(p - n + i_1)\mu(i_1,\ldots,i_k) - (i_1 - i_2)\mu(i_2,\ldots,i_k) \cdots\cdots - (i_{k-1} - i_k)\mu(i_k) \tag{4.23}$$

where $\mu(i_1,\ldots, i_k)$ denotes the number of the sequences $(j_1,\ldots,j_k)$ of integers which satisfy the following criteria :

$$\begin{aligned} & j_1 \geq j_2 \geq \cdots \cdots \cdots j_k \geq 0 \\ & i_r \geq j_r \end{aligned} \tag{4.24}$$

for all $s \leq r \leq k$, $j_s > 0$. Hence, the singularity sets $\Sigma^{i_1,\ldots,i_k} \subseteq J^k(n, p)$ are called *Boardman submanifolds* of $J^k(n, p)$.

THEOREM 4.8 *([1] , Section 5.8.1)*

Any stable mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is generic (in the sense of Boardman) .

4.8 *THE PROCEDURE OF ANALYSIS*

Study of bifurcation involves the analysis of the solutions in the neighborhood of singular points. The solution structure about a singular point is the property of the type of singularity of the map about that point. Further analysis is based on this consideration.

About a singular point, the map can be modified keeping non-linear characteristics invariant. Conditions for such approximation are specified by contact invariance and subsequently Boardman symbols.

Satisfying these conditions, an equivalent map can be constructed on minimal neccesary information. The bifurcation related phenomena can be studied on this map and a correspondence can be found between the variation of parameters and the type of resulting bifurcation diagram. Such analysis also helps in providing good guesses for plotting of exact bifurcation diagram about that point.

For one dimensional case, the maps for which Taylor series expansions about a point in variable domain have coefficients of all the terms upto second order as zero, show bifurcation characteristics similar to those of ,

$$0 = x^3 + \alpha x + \beta \qquad (4.25)$$

This is a minimal representation of solution structure around the singular point. The solutions can be represented in three dimensions. Presence of any more parameters do not lead to any system behavior which is not explained by this equation. For general r-parameter (λ) deformation of the original Taylor series expansion , same representation holds. The bifurcation behavior can still be explained by equation (4.25), execpt that

scales on (x, α, β) are parameterised by $(\lambda_1, \ldots, \lambda_p)$. Due to lesser number of dimensions required to present the bifurcation characteristics, the manifold described by the equation may be plotted. This provides an aid to predict the type of bifurcation that are possible with general variation of $(\lambda_1, \ldots, \lambda_p)$.

The set of evolution equations for vibration of the string can be represented as a mapping

$$\begin{aligned} f:\ \Lambda \times \mathbb{N} &\longrightarrow \mathbb{V} \\ \mathbb{N} &\subseteq \mathbb{R}^4 \\ \Lambda &= \mathbb{R}^5 \\ \mathbb{V} &= \mathbb{R}^4 \end{aligned} \tag{4.26}$$

The mapping is expanded about the given singular point . The map is considered to be a 5-parameter deformation of $f \in \mathcal{E}^0_{4,4}$. Hence, the initial truncated segment of the Taylor series upto order k, $J^k f$, can now be taken to be as an element of $J^k(4, 4)$ for a fixed value of λ. For purpose of representation , however, an augmented map is considered which is a 5-parameter unfolding of the map about the singular point,

$$F : \Lambda \times \mathbb{N} \longrightarrow \Lambda \times \mathbb{V} \tag{4.27}$$

given by $(\lambda, x) \longrightarrow (\lambda, f(\lambda, x))$. The map F is smooth and element of $\mathcal{E}^0_{9,9}$. Introducing a-equivalence on this set theorem (4.1) can be applied. The theorem states that if the germ is of rank p then there exists an invertible germ $h: (\mathbb{R}^9, 0) \longrightarrow (\mathbb{R}^9, 0)$ such that $G = F \circ h$ is a p-parameter unfolding of germ , g, of rank zero.

The rank of the germ depends on the rank of its Jacobian. As 5 parameters are already defined, rest of the parameters are defined by the rank of the map $f_0: \mathbb{N} \longrightarrow \mathbb{V}$. The map g is the restriction of the mapping to the null space $\mathbb{Q}$ of $\mathbb{N}$. $\mathbb{P}$ is taken

as compliment of $\mathbb{Q}$ in $\mathbb{N} \times \Lambda$. We therefore have,

$$G : \mathbb{P} \times \mathbb{Q} \longrightarrow \mathbb{P} \times \mathbb{V}$$
$$G(p, q) \longrightarrow (p, g(p,q)). \qquad (4.28)$$

As F and G are a-equivalent, theorem (4.2) helps in looking for another a-equivalent p-parameter unfolding using k-equivalence on g_o . Thus the problem of studying 5-parameter deformation of f is transformed to p-parameter deformation of g_o.

This part of the operation is performed in program **LYAPUNOV**. The program **ANALYSIS** contains routines based on theory discussed in the previous sections to represent the map in minimal dimensions. The methodology of such reduction is to construct an a-equivalent graph in the neighborhood of singular point.

To G a germ $D_G : (\mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^q, 0) \longrightarrow \mathbb{R}^m$ is associated given by $(\alpha, w, x) \longrightarrow -w + g(\alpha, x)$. Hence, $D_G = 0$ defines the graph of the equation given by, $w = g(\alpha,x)$.

Thus, D_G is an p+m parameter submersive deformation. In this case $\Pi_{D_G} : (V_{D_G}, 0) \longrightarrow (\mathbb{R}^{p+q}, 0)$ can be identified with g.

Next step involves an assumption that D_G is a k-versal deformation, i.e., there exist small parameter deformations for which D_G is transverse to all the smooth manifolds of $J^k(n, p)$ which define the k-equivalence of g .

Theorem (4.3) is an observation that if two germs D_G and $D_{G'}$ are k-versal p+m parameter deformations of germs g_o and g'_o of rank zero and, g_o and g'_o are k-equivalent, then Π_{D_G} and $\Pi_{D_{G'}}$, i.e., G and G' are a-equivalent.

At this stage the aim of the operations becomes clear, The search is for k-universal $D_{G'}$ such that G and G' are a-equivalent. To find k-universal $D_{G'}$ there is a need to study the k-equivalence

class defined by $\mathcal{g}_o$.

The statement of theorem (4.6) points out that the Boardman symbol of a germ is contact-invariant. Hence, to classify the germ $\mathcal{g}_o$ under k-equivalence , there is need to compute the Boardman symbol for $\mathcal{g}_o$ and to set up the set of equations by which the germ $\mathcal{g}_o$ is a member of that Boardman submanifold of $J^k(n,p)$.

4.9 *BOARDMAN SYMBOL FOR A GIVEN MAP OF RANK ZERO*

The Boardman symbol of a germ corresponds to Thom's singularity set. The procedure seeks to construct the set of equations by which $J^k f$ is a member of some Boardman submanifold in $J^k(4,4)$. The map $\mathcal{g}_o : \mathbb{R}^q \rightarrow \mathbb{R}^q$ is restriction to the q-dimensional null space of the map $f_o : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ and has same Boardman symbol. Proceeding on similar lines as in Chapter 3, Boardman symbol is computed.

Recalling the attempt in the previous chapter to look for singular point or bifurcation point, by satisfying conditions on the characteristic polynomial such as equating coefficients of characteristic polynomial to zero, the methodology based on Boardman's theory can be considered to be a recursive use of the same procedure.

Given a smooth singular map with nil-potent Jacobian (*i.e.*, the rank = 0), $\mathcal{g}_o : \mathbb{R}^q \rightarrow \mathbb{R}^q$, $\mathcal{g}_o(x) = v$, following equations may be satisfied,

$$\left.\begin{array}{ll} \mathcal{g}(x) & = 0 \\ C_o\ P_q(\mathcal{g}_o(x)) & = 0 \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \\ C_l\ P_q(\mathcal{g}_o(x)) & = 0 \end{array}\right\} \begin{array}{l} \text{I} \\ \text{II} \end{array} \qquad (4.29)$$

Where the expressions on the left hand side of the equation set (II) are the coefficients of the characteristic polynomial of the q x q Jacobian matrix of $g_o(x)$. This set of expressions is obtained by procedure GET_AUX_EQ. It is to be noted that some of these equations have already been satisfied in search of bifurcation point. The criteria defining the k-equivalence are contructed by sequential transfer of equations from set (II), to an another set (III), satisfying the condition that the linear terms in x_λ are absent. The set (III) is originally empty.

The methodology in creating set (III) , is as follows. If $\iota \leq q$ and $C_\iota P_q(g_o(x))$ has linear terms in it , addition of more equations to set (III) is stopped. By a transformation on the variable domain a variable (x_q) is set up to denote that expression by a required linear invertible transformation and the expression is stored in set (III). The set (I) is replaced by the set (II) , with the restriction ($x_q = 0$) and next stage of the operations are performed on this set with q-1 equations and the process is repeated until all variable directions are so determined. The maximum of index ι in each stage constitute the Boardman symbol. The equations in set (III) define the Boardman submanifold.

4.10 *MINIMAL UNFOLDING OF THE GERM OF RANK ZERO*

Operation performed as above can be repeated for the map $g(\lambda, z) = v$. The z in this equation are the ones transformed by the above procedure. Each of the variables (z), represents a linear equation in the set (III). The parameterisation of these equations by Λ can again be represented by a corresponding translation on $\mathcal{X}$. The constant terms which occur in the original map (4.26) due to small variations in λ , are taken care of by

translation in $\mathbb{V}$. Thus we have ,

$$\begin{aligned} \mathcal{g}(\lambda,\ y) &= u \\ u &= u(v,\ \lambda) \\ y &= y(z,\ \lambda) \end{aligned} \qquad (4.30)$$

The procedure K_EQ_RED performs the operations mentioned above given a germ of rank 0.

Putting u = 0 and y = 0 in the equation set (III), some equation are obtained in terms of λ. Parameters are also transformed based on these equations , say $\alpha = \alpha(\lambda)$. Dimension of α depends on the number of equations in the set at y = 0 and u = 0. This in turn depends on co-dimension of the Boardman submanifold in the jet-space. These reductions are performed by the routine UNFOLDING_DIR in the program ANALYSIS.

By making transformations in the domain of D_G it can be claimed that $D_G,(u,\ \alpha,\ y)$ is a k-universal deformation , as α is the minimal set of parameters that can describe the versal unfolding.

If α can be represented as to span a subset of $\mathbb{R}^4 - \Omega$, then it can be claimed that the transformed map is an a-trivial representation in $J^k(4,4)$. The affect of small variations of parameter for such a representation is that of translation of the graph in $\mathbb{X} \times \mathbb{V}$.

4.11 *CHARACTERISTICS OF A-STABLE GERMS*

If an a-stable representation is found then graph (f) is representable in four dimensions or even less. Thus for these cases variation of parameters does not change bifurcation

characteristics. If the map $g(x, \lambda) = v$ has an a-trivial representation , then it is possible to reduce the map to following form.

$$\begin{aligned} v' &= g(x') \\ v' &= v'(v, \lambda) \\ x' &= x'(x, \lambda) \end{aligned} \qquad (4.31)$$

where $v'_v (v, \lambda)$ and $x'_x(x, \lambda)$ are invertible.

Thus, it is feasible to plot the graph(g') in $V' \times X'$. Given a value for λ , v' can be computed using equation (4.31.ii) taking v = 0. For that value of v' the possible solutions (x') can be obtained and since,

$$x'(x, \lambda) = x' \qquad (4.32)$$

is invertible, all solutions (x) for that value of λ can be obtained. In this way an approximate bifurcation diagram for a given perturbation in the conditions can be plotted with respect to any combination of parameters. Having obtained the solutions the dynamic stability on each of them may be predicted easily.

Chapter 5

CONCLUSIONS

The study of large amplitude oscillations of stretched strings provides a representative view of the non-linear phenomena which occur in actual systems. For the case of stretched strings such phenomena, *e.g.*, whirling and jump, occur near linear resonance frequencies. These result from the exchange of stability that occurs between the solutions depicting different motion characteristics.

The equations governing the motion of string are formulated in Chapter 2 with a view to represent these phenomena on a larger time scale. The important assumptions made in the process of the formulation are that the wave lengths of the modes are comparatively much larger than the amplitudes in the transverse directions (2.10), and the velocity of sound in the material of the string is much greater than the velocities of the transverse modes (2.17). The second assumption reduces the number of the differential equations from three to two, equations (2.25) and (2.26).

These equations represent the system to a fair degree of accuracy. The governing equations, however, become singular for a forcing frequency in the neighborhood of the linear natural frequencies. For this reason the course of analysis is to first expand the equations about the solutions of linearised system. Hence, the spatial displacements are approximated by those of the linear modes and the method of multiple scales is applied. This

variables in the kernel. A step in these directions and the corresponding steps in those spanning the compliment, provides a good guess for the program CONTINUATION.

The program CONTINUATION is used to plot the bifurcation diagram for a given parameter. This procedure relies on minimising a functional (3.18) to obtain the solutions as they vary with the parameter. The solutions so obtained are plotted with respect to a given parameter directions.

These programs were run for a representative set of values. A bifurcation diagram was plotted for the given conditions with respect to pertubation in the frequency of the forcings from the linear resonance frequency, (Fig 3) and (Fig 4) . A bifurcation diagram was also plot with respect to amplitude of support movement (Fig 5).

The bifurcation diagrams, plot using such algorithms, do not provide much detail about bifurcation characterstics of multi-parameter systems for following reasons :

(i) All possible invertible solutions arising due to variation of a parameter may not be obtained.

(ii) The continuation scheme is based on minimisation of a functional, $f^T f$. As the determinant of the jacobian in the neighborhood of singular point is small, variation of parameter induces a change of high order ($\frac{\Delta \langle o \rangle}{\langle o \rangle}$)in the determinant, restricting the corresponding step size in the variable domain.

(iii) These procedures are not capable depicting bifurcation diagrams in which singular point is absent. Such bifurcation diagrams are important to study the occurance of jump phenomena.

5.1 *THE AIM OF LOCAL ANALYSIS*

The program **ANALYSIS** was coded to aid these procedures in ploting of approximate bifurcation diagrams for any set of conditions and with respect to any given parameter direction, so as to predict the types of bifurcations possible. The technique followed, is based on the theories and procedures discussed in the Chapter 4. The program, given an unfolding of a germ of rank zero, relies on the equivalence conditions defined by the Boardman symbol to look for an approximate trivial unfolding.

The assumption made in the process of such computations is that the given mapping is a k-versal unfolding. The pseudo parameters can be added to the system to make the unfolding k-versal and, on a later stage equated to zero. The program aims to map the given unfolding to one of the a-stable representations of mappings $\mathbb{R}^9 \longrightarrow \mathbb{R}^4$. The graph of this representation can be plotted.

The graph so defined corresponds, with invertible transformations on the coordinates, *i.e.* range, domain and parameters, to all possible bifurcation diagrams in the neighborhood of the singular point. Another characteristic of such map is that the bifurcation behavior, with respect to a given parameter and for given non-zero values of rest of the parameters, does not change with small change in parameters and all solutions are invertible.

Obtaining a-stable representation of the mapping under a-equivalence relation reduces the problem of analysis to a large extent. The classification efforts are aided by the fact that there exist only seven types of a-stable representations for mappings between 4-dimensional spaces, as listed in *([1], Section 5.8.6)*. Study of such singularities around (x = 0) and (λ = 0) can completely unravel all the possible bifurcations of the stationary solutions that can occur in the system.

The possible bifurcation characteristics for a given singular point correspond to the Boardman symbol. Some simple bifurcation properties possible of generic maps $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$ are listed below.

(i) Σ^0 $\qquad y_i = x_i \qquad (i = 1, 2, \cdots, 4)$ $\qquad$ (4.33)

This symbol denotes a regular point and no bifurcation can ever be observed for cases when the map around a point is representable in this form (4.33).

(ii) $\Sigma^{1,0}$ $\qquad y_i = x_i \qquad (i = 1, 2, 3)$

$\qquad\qquad\quad y_4 = x_4^2$ $\qquad$ (4.34)

The maps a-equivalent to above map can show bifurcation characteristics in which two solutions emerge from the point. Bifurcations represented by a pair of intersecting lines are represented by forms of this type. For the germs of this class the solution may disappear for some value of parameters.

(iii) $\Sigma^{1,1,0}$

$$y_i = x_i \qquad (i = 1, 2, 3)$$
$$y_4 = x_4^2 + x_1 x_4 \tag{4.35}$$

The singular points which have the form (4.35) associated with them are capable of displaying pitch-fork bifurcations and the bifurcation characteristics arising due to breaking of the pitch-fork. The singular point obtained for (Fig 3) is one of this type. It is to be noted that for such cases at least one solution is always present as against previous case when for some value of parameters solution could have disappeared. These cases are important for general stability analysis as phenomena of jump can first be observed at only such points.

(iv) $\Sigma^{1,1,1,0}$

$$y_i = x_i \qquad (i = 1, 2, 3)$$
$$y_4 = x_4^4 + x_1 x_4 + x_2 x_4^2 \tag{4.36}$$

(v) $\Sigma^{1,1,1,1,0}$

$$y_i = x_i \qquad (i = 1, 2, 3)$$
$$y_4 = x_4^5 + x_1 x_4 + x_2 x_4^2 + x_3 x_4^3 \tag{4.37}$$

These cases represent more complex solution organisation around singular points of the respective types, from which upto four (4.36) or five (4.37) solutions may emerge on variation of certain parameters.

(vi) $\Sigma^{2,0}$

$$y_i = x_i \qquad (i = 1, 2)$$
$$y_3 = x_3 x_4$$
$$y_4 = x_3^2 + x_4^2 + x_1 x_3 + x_2 x_4 \tag{4.38}$$

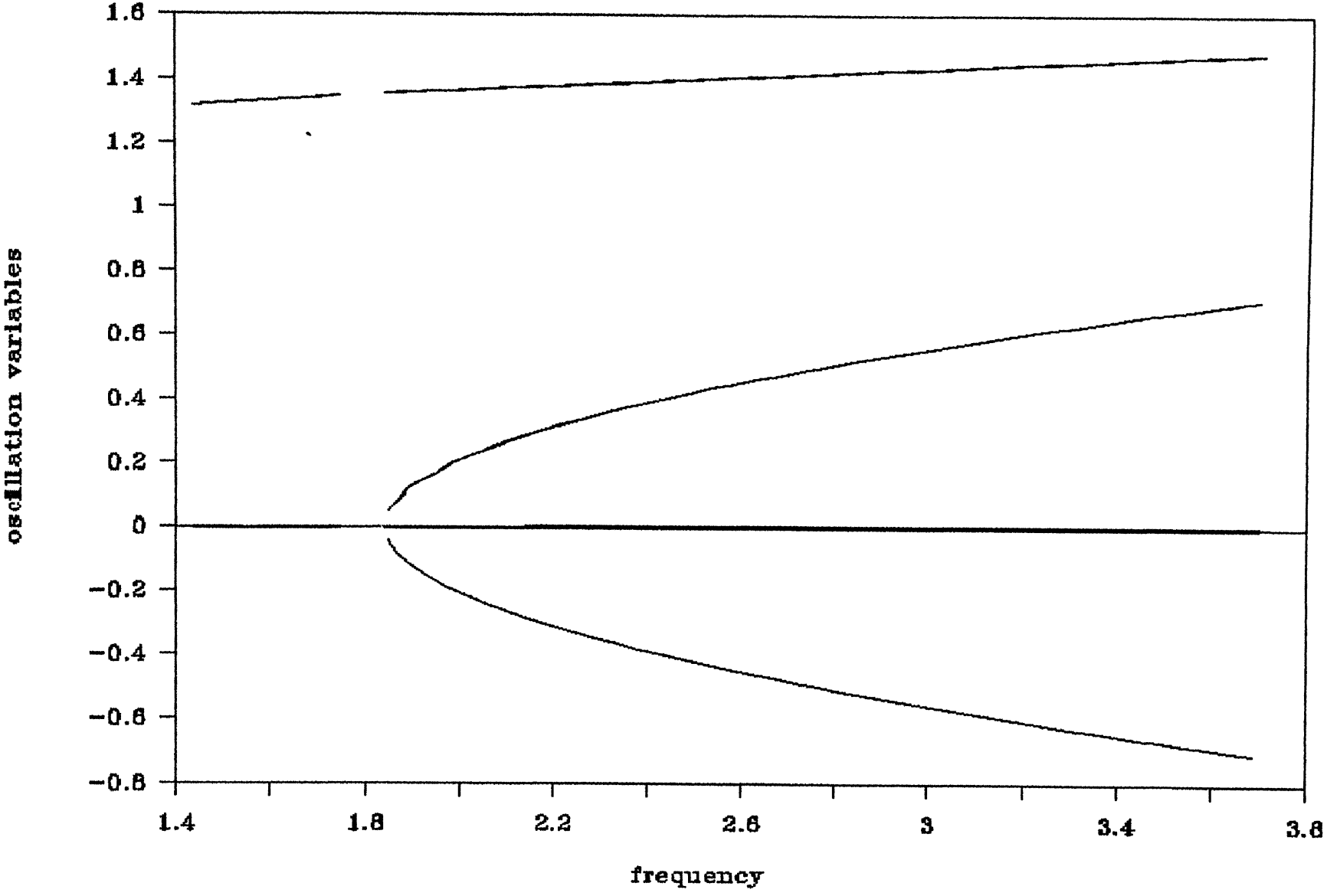

Fig. 3

	x_1	x_2	x_3	x_4	μ	σ	ξ	k	ψ
Singular Point	1.357	0.003	0.0	0.0	0.01	1.84	0.0	5.0	0.0

$\Sigma^{1,1,0}$

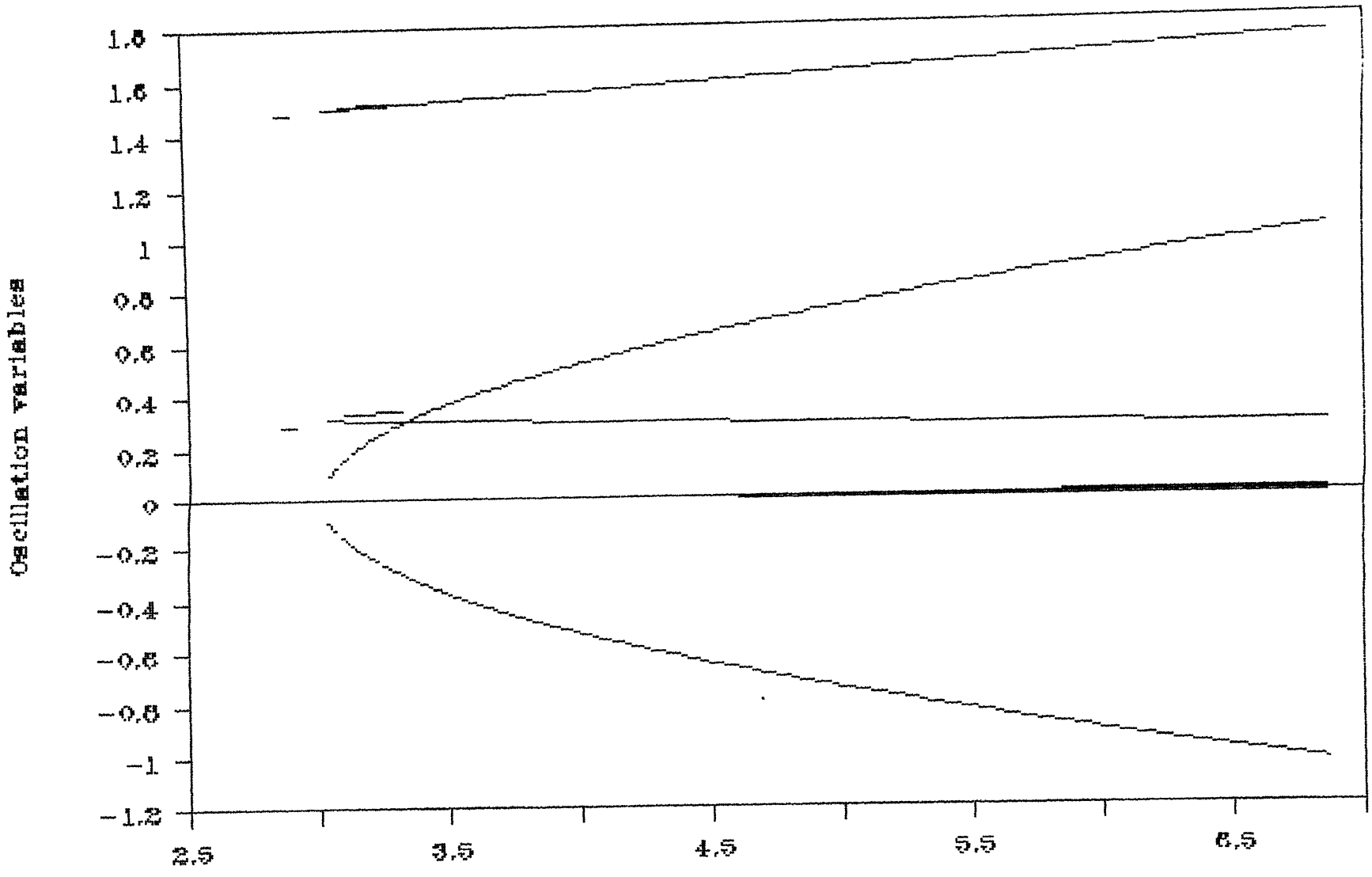

Fig. 4

	x_1	x_2	x_3	x_4	μ	σ	ξ	k	ψ
Singular Point	1.4966	0.310	0.0	0.0	0.01	3.0	1.051	5.0	116.68°

$\Sigma^{3,2,1,0}$

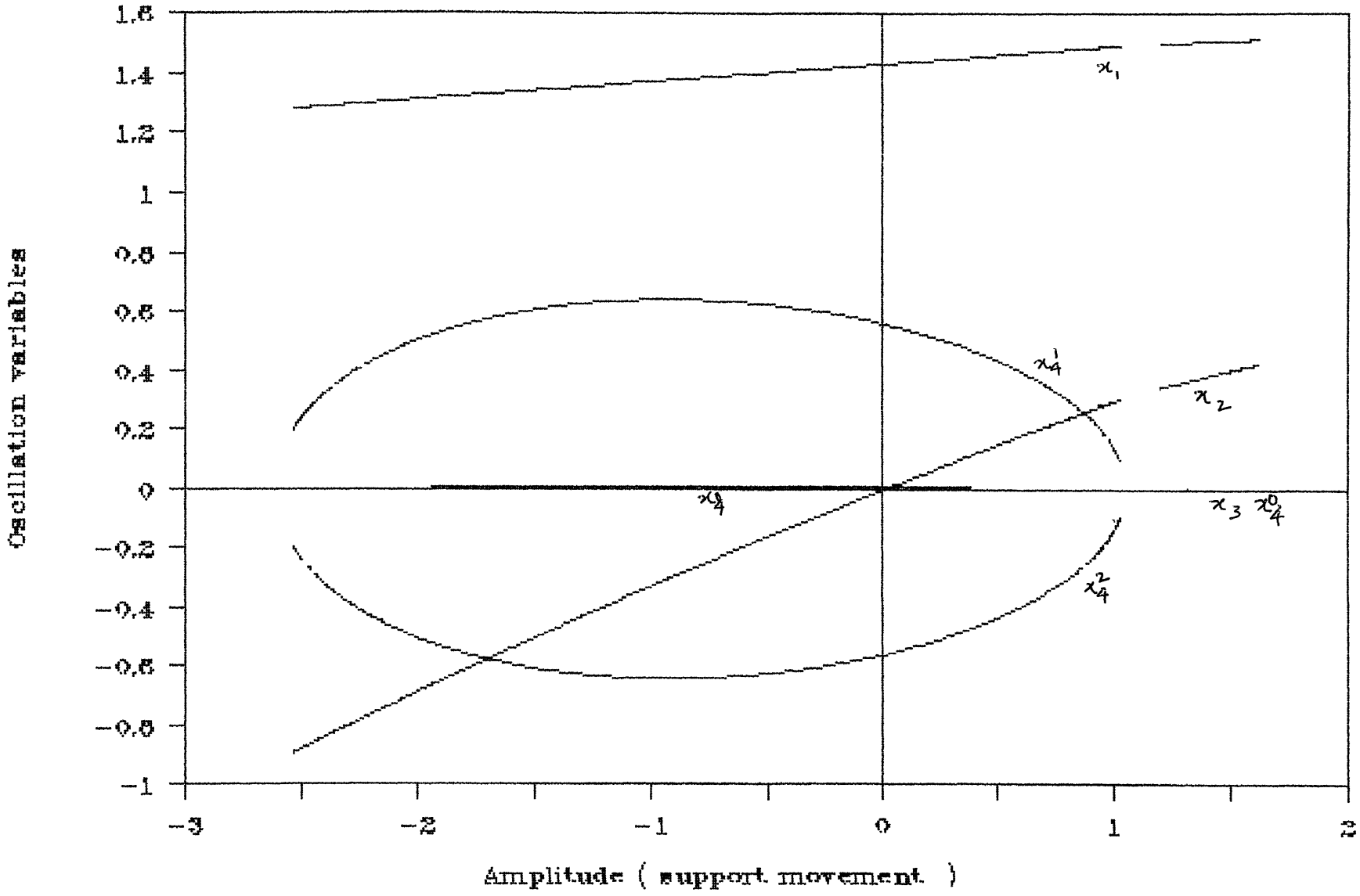

Fig. 5

	x_1	x_2	x_3	x_4	μ	σ	ξ	k	ψ
Singular Point	1.4966	0.310	0.0	0.0	0.01	3.0	1.051	5.0	116.68°

$\leq 3,2,1,0$

$$y_i = x_i \qquad (i = 1, 2)$$

Coll) $\Sigma^{2,0}$ $$y_3 = x_3 x_4 \qquad (4.39)$$

$$y_4 = x_3^2 - x_4^2 + x_1 x_3 + x_2 x_4$$

Both these forms represent more diverse bifurcation phenomena than the previous cases. Form (4.38) represents the solutions arising from intersection of a hyperbola and a circle in a plane defined by x_3 and x_4 . The expansions about singular points which are a-equivalent to form (4.39) show bifurcations arising from intersection of two hyperbolas.

The program ANALYSIS was coded for the purpose of obtaining such representations for given expansions and analysing them. However satisfactory runs were not possible due to irregularity and unavailability of computational facilities and time.

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